Towards a bijective enumeration of spanning trees of the hypercube

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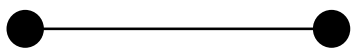
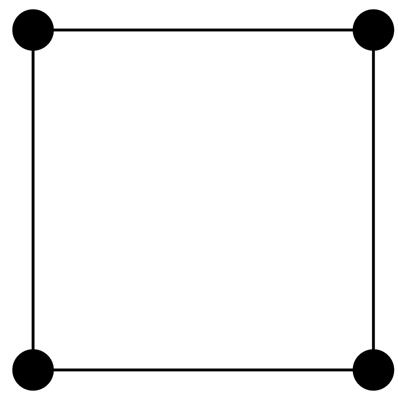
Full paper
"Factorization of some weighted spanning tree enumerators"

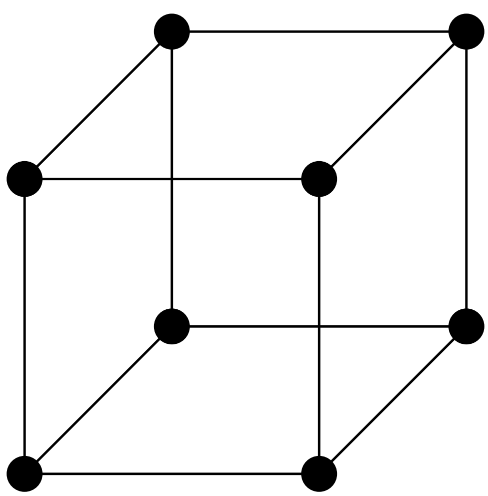
at `http://math.umn.edu/~martin/math/pubs.html`

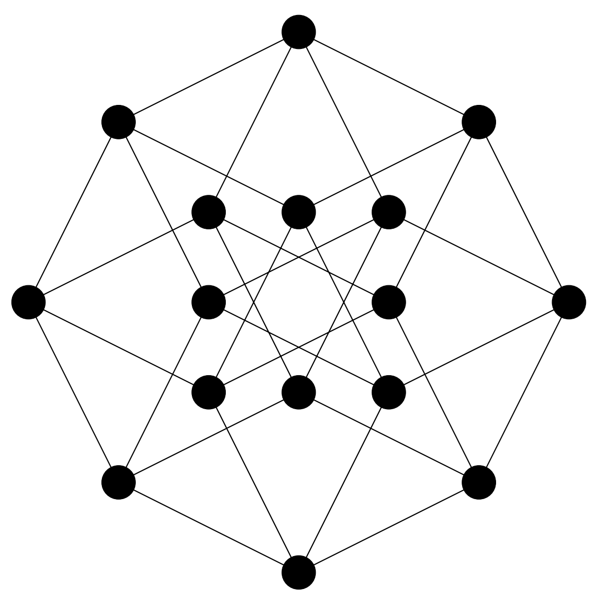
The hypercube Q_n

$$
\begin{aligned}
V(Q_n) &= \{v = v_1 v_2 \ldots v_n : \ v_i \in \{0,1\}\} \\
E(Q_n) &= \{vw : \ v_i = w_i \text{ for all but one } i\}
\end{aligned}
$$









Spanning trees of Q_n

$$\begin{aligned}\mathrm{Tree}(G) &= \{\text{spanning trees of a graph } G\}\\ \tau(G) &= |\mathrm{Tree}(G)|\\ [n] &= \{1,2,\ldots,n\}\end{aligned}$$

Theorem 0 (Stanley, Enumerative Combinatorics, vol. 2, p. 62)

$$\tau(Q_n) = \prod_{\substack{S\subset[n]\\|S|\geq 2}} 2|S| = 2^{2^n-n-1}\prod_{k=1}^{n} k^{\binom{n}{k}}.$$

$$\begin{aligned}\text{E.g., } \tau(Q_3) &= 2|\{1,2\}| \cdot 2|\{1,3\}| \cdot 2|\{2,3\}| \cdot 2|\{1,2,3\}|\\ &= 4\cdot 4\cdot 4\cdot 6 = 384.\end{aligned}$$

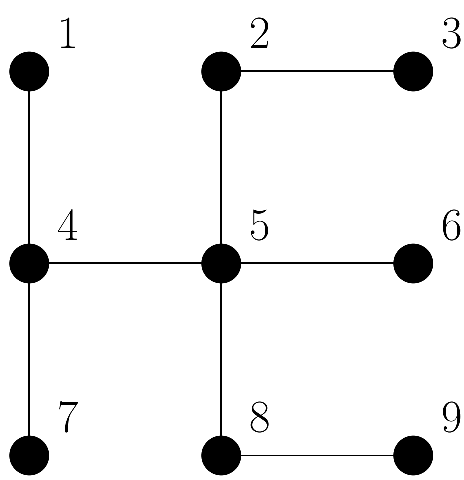
Bijective proof??

The model: K_n and the Prüfer code

K_n = complete graph on n vertices

Cayley's Formula: $\tau(K_n) = n^{n-2}$

Prüfer code: $\mathrm{Tree}(K_n) \xrightarrow{\text{bijection}} [n]^{n-2}$



$\longleftrightarrow$

Prüfer code
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- $\deg_T(i) = 1 + \text{number of } i\text{'s in Prüfer code of } T$

Cayley-Prüfer Formula:

$$\sum_{T \in \mathrm{Tree}(K_n)} x_1^{\deg_T(i)} \cdots x_n^{\deg_T(n)} = x_1 \cdots x_n \, (x_1 + \cdots + x_n)^{n-2}$$

Weighted enumeration and bijections

- Suppose that you know Cayley's formula $\tau(K_n) = n^{n-2}\ldots$
  $\ldots$ and can prove it using the Matrix-Tree Theorem$\ldots$
  $\ldots$ but are looking for a *bijective* proof.

- Knowing the Cayley-Prüfer Formula

$$\sum_{T \in \mathrm{Tree}(K_n)} x_1^{\deg_T(i)} \cdots x_n^{\deg_T(n)} \quad = \quad x_1 \cdots x_n \, (x_1 + \cdots + x_n)^{n-2}$$

might be an important clue, enabling you to reproduce the Prüfer code (or a similar bijection).

- **<u>Goal:</u>** Do the same thing for Q_n by finding a weighted analogue of the formula

$$\tau(Q_n) \quad = \quad \prod_{\substack{S \subset [n] \\ |S| \geq 2}} 2|S|$$

Weighted enumeration of spanning trees of Q_n

- Assign a monomial weight $\mathrm{wt}(e)$ to each edge $e \in Q_n$,

  define $\quad \mathrm{wt}(T) = \prod_{e \in T} \mathrm{wt}(e) \quad$ for $T \in \mathrm{Tree}(Q_n)$,

  and consider the generating function

$$\sum_{T \in \mathrm{Tree}(Q_n)} \mathrm{wt}(T).$$

First attempt: Keep track of vertex degrees (à la Prüfer).

Weight each edge $vw \in E(Q_n)$ by

$$\mathrm{wt}(vw) \;=\; y_v y_w$$

so that

$$\mathrm{wt}(T) \;=\; \prod_{v \in V(Q_n)} y_v^{\deg_T(v)}$$

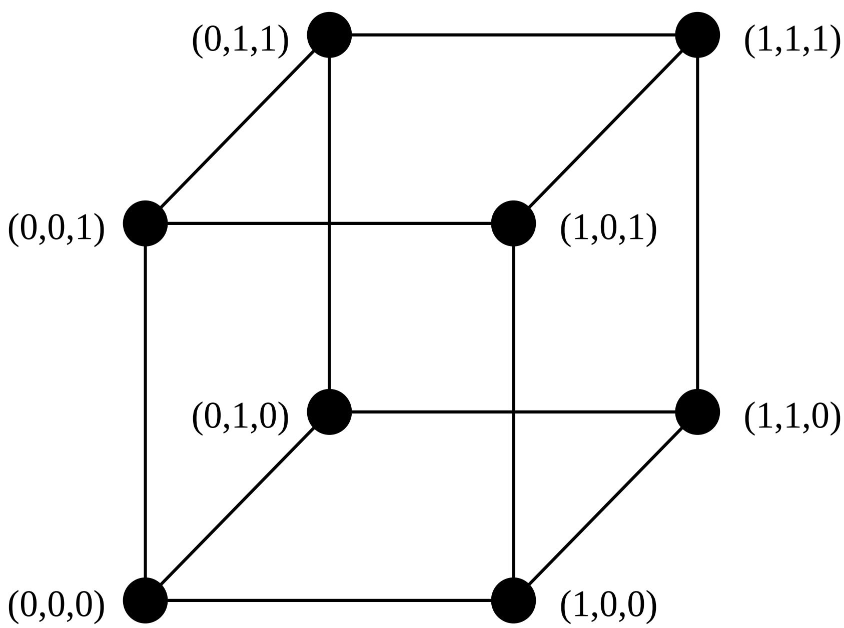
- Unfortunately, this does not factor nicely. E.g., for $n = 3$, it is

  $x_{000} \cdot x_{001} \cdots x_{111} \cdot$ (some irreducible degree-6 nightmare).

Directions of edges

- Weight each edge vw by q_i, where $i = \mathrm{dir}(vw)$ is the unique index for which $v_i \neq w_i$. So

$$\mathrm{wt}(T) \;=\; q^{\mathrm{dir}(T)} \;=\; \prod_{i=1}^{n} q_i^{|\{\text{edges of } T \text{ in direction } i\}|}$$



Theorem 1

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} \quad = \quad 2^{2^n-n-1}\, q_1 \cdots q_n \prod_{\substack{S \subset [n] \\ |S| \geq 2}} \left(\sum_{i \in S} q_i \right)$$

Decoupled vertex degrees

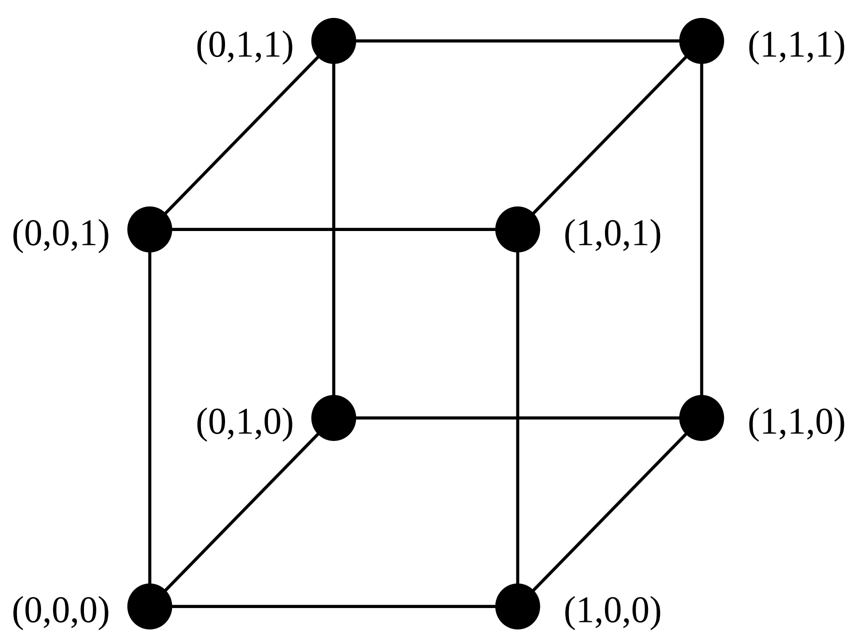
- For each edge $e = vw$ not in direction i,

$$\text{either} \qquad v_i = w_i = 0 \qquad \text{or} \qquad v_i = w_i = 1.$$

Weight e by x_i or x_i^{-1} accordingly. E.g., for $e = \{\mathbf{0}10, \mathbf{1}10\}$,

$$\text{wt}(e) \;=\; q_{\text{dir}(e)} x^{\text{dd}(e)} \;=\; q_1 x_2 x_3^{-1}.$$

- Equivalently, record which $Q_{n-1} \subset Q_n$ the edge e belongs to.



The main result

Theorem 2

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)} \quad = \quad q_1 \ldots q_n \prod_{\substack{S \subset [n] \\ |S| \geq 2}} \underbrace{\left(\sum_{i \in S} q_i (x_i^{-1} + x_i) \right)}_{f_S}$$

where $\quad q^{\mathrm{dir}(T)} = \prod_{e \in T} q_{\mathrm{dir}(e)}, \qquad x^{\mathrm{dd}(T)} = \prod_{e \in T} x^{\mathrm{dd}(e)}.$

Compare Theorem 1:

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} \quad = \quad 2^{2^n - n - 1}\, q_1 \cdots q_n \prod_{\substack{S \subset [n] \\ |S| \geq 2}} \left(\sum_{i \in S} q_i \right)$$

and Theorem 0:

$$\tau(Q_n) \quad = \quad \prod_{\substack{S \subset [n] \\ |S| \geq 2}} 2|S|$$

Sketch of the proof

Weighted Matrix-Tree Theorem

Let $L = (L_{vw})_{v,w \in V(G)}$ be the weighted Laplacian:

$$L_{vw} = \begin{cases} 0 & v \neq w \text{ and } vw \notin E(G) \\ -\operatorname{wt}(vw) & vw \in E(G) \\ \displaystyle\sum_{e \ni v} \operatorname{wt}(e) & v = w \end{cases}$$

Then $\sum_{T \in \mathrm{Tree}(G)} \operatorname{wt}(T) = \det \hat{L}$, where $\hat{L}$ is obtained by deleting the vth row and vth column of L.

Identification of Factors Lemma (Krattenthaler)

$f \mid \det \hat{L} \iff \hat{L}$ has a nullvector in $\mathbb{Q}[q,x]/(f)$.

- Use a computer algebra package (e.g., Macaulay) to compute "witness" nullvectors for factors $f = f_S$
- Experimentally, the witnesses have a nice form, reducing the proof to calculation
- Same method can be used for threshold graphs (specializing a result of Remmel and Williamson) and products of K_n's