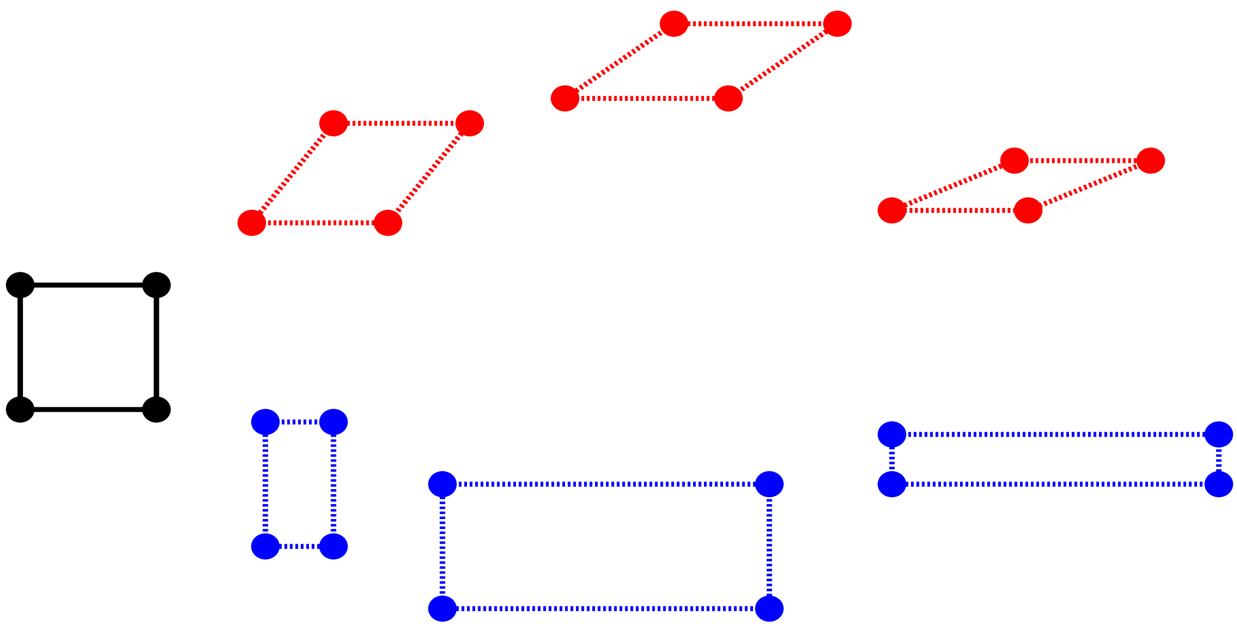
Rigidity Theory for Matroids



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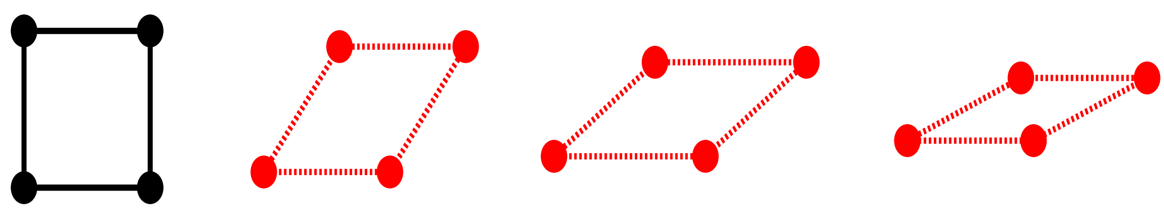
Rigidity Theory for Graphs

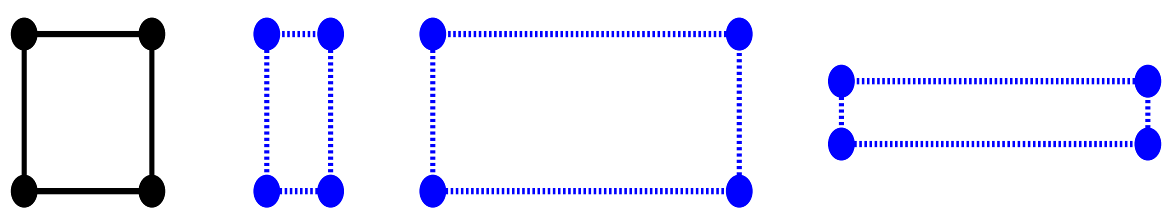
Framework for a graph $G = (V, E)$ in $\mathbb{R}^d$:

$$\text{joints} \longleftrightarrow \text{vertices}$$
$$\text{bars} \longleftrightarrow \text{edges}$$

Pivoting framework: bars are fixed in length, but can pivot around joints



Telescoping framework: bars are attached to joints at fixed angles, but are allowed to change in length



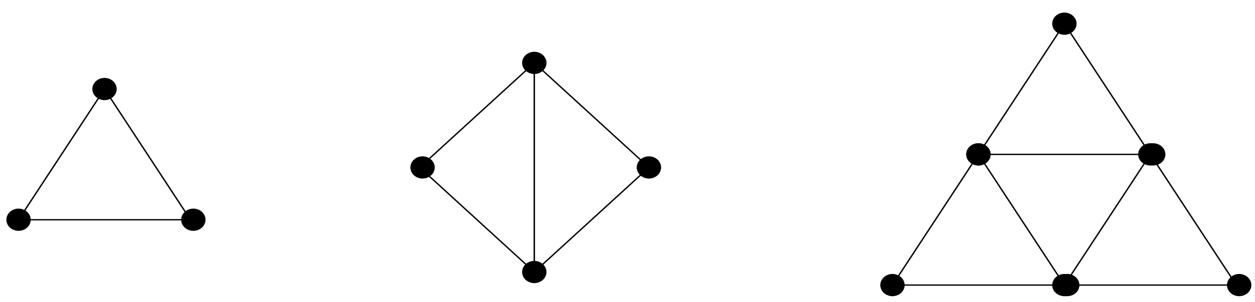
Problem: When is a framework in $\mathbb{R}^d$ rigid?

Examples of Rigid and Flexible Graphs

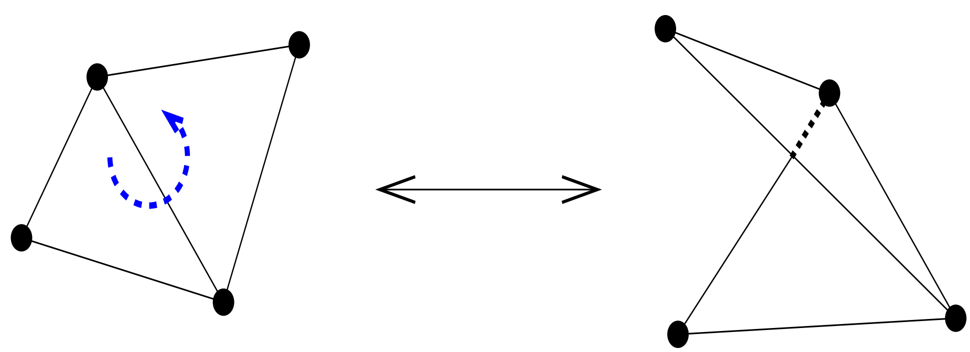
- A graph is 1-rigid if and only if it is connected.

- Every d-rigid graph is d-connected, and in particular has minimum degree $\geq d$.

- Every triangulation is 2-rigid.



- Triangulations are typically not 3-rigid.



Matroids

• A **matroid independence system** M on a finite ground set E is a collection of subsets of E such that...

$$\begin{array}{ll} (1) & \emptyset \in M; \\ (2) & I \subset J,\ J \in M \quad \implies \quad I \in M; \\ (3) & I, J \in M,\ |I| < |J| \quad \implies \quad \exists e \in J - I:\ I \cup e \in M. \end{array}$$

A matroid can be described equally well by any of the following data:

Bases	(maximal independent sets)
Circuits	(minimal dependent sets)
Rank function	$r(A)$ = size of maximal ind't subset of A
Closure operator	$\bar{A} = \{e :\ r(A \cup e) = r(A)\}$

Linear matroid: E = set of vectors
M = {linearly independent subsets}

Graphic matroid: E = edges of a graph
M = {acyclic edge subsets}

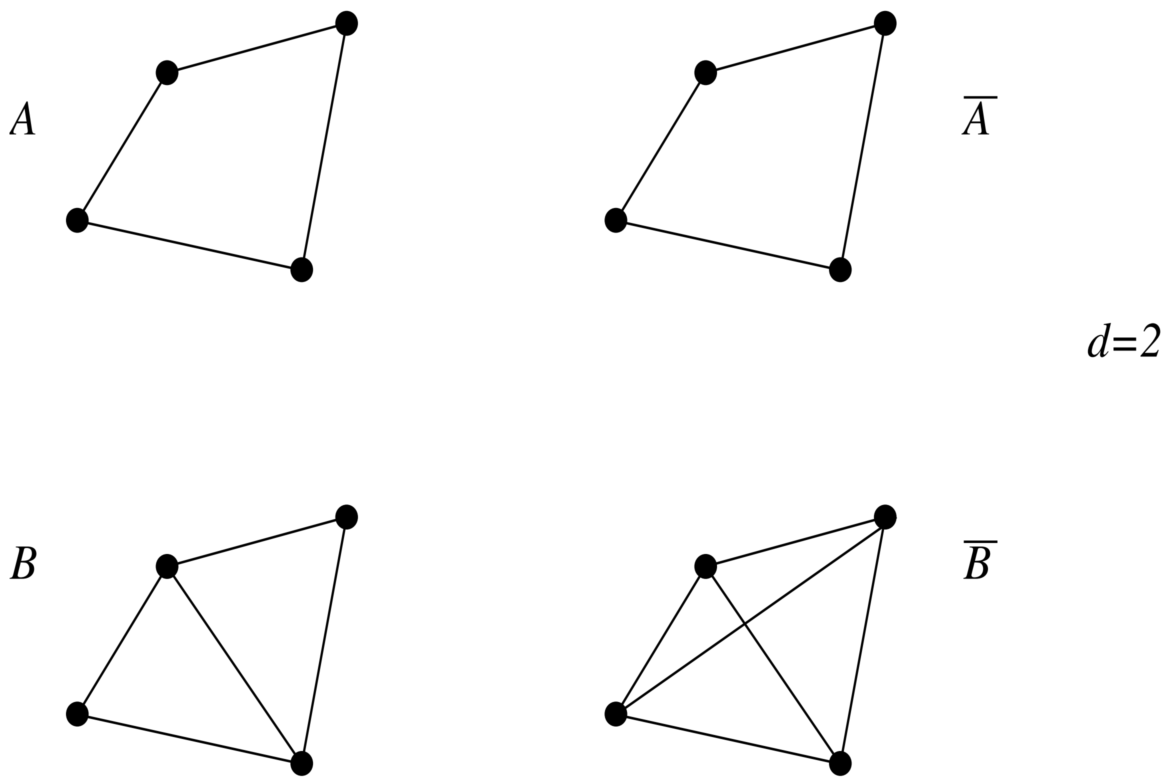
Tutte polynomial of M (an incredibly nice invariant!):

$$T_M(x, y) \;=\; \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

The d-Rigidity Matroid of a Graph

Let $G = (V, E)$ be a graph and $d \geq 2$ an integer. Define the **d-rigidity matroid $\mathcal{R}^d(G)$** on E by the closure operator

$$\overline{F} := \{\text{edges whose length in every generic pivoting framework in } \mathbb{R}^d \text{ is determined by the lengths of the edges in } F\}$$



- Replacing "length" with "slope" gives the **d-slope matroid** (or **d-parallel matroid**), denoted $\boldsymbol{\mathcal{S}^d(G)}$.

Representing the d-Rigidity Matroid

$\mathcal{R}^d(G)$ can be represented by the **d-rigidity matrix** $R = R^d(G)$

- R has $|E|$ rows and $d|V|$ columns
  Rows of $R \longleftrightarrow$ edges
  Columns of $R \longleftrightarrow$ coordinates of vertices in $\mathbb{R}^d$
  Entries of R are polynomials in $d|V|$ variables

- Right nullvectors of R (syzygies among columns)
  = infinitesimal motions of vertices that preserve all edge lengths

$$G \text{ is } d\text{-rigid} \iff \text{right nullspace} = \{\text{rigid motions of } \mathbb{R}^d\}$$
$$\iff \operatorname{rank} R = d|V| - \binom{d+1}{2}$$

- Left nullvectors of R (syzygies among rows)
  = polynomial constraints ("stresses") on edge lengths
- $r(F)$ = rank of corresponding row-selected submatrix of R

$$G \text{ is } d\text{-rigidity-independent} \iff \text{left nullspace} = 0$$
$$\iff \mathbb{R}^d(G) = 2^E$$

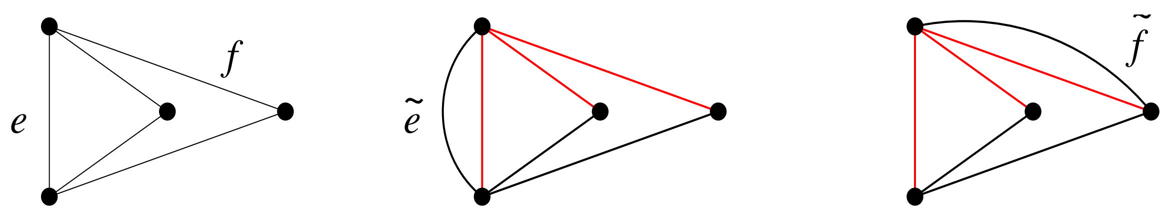
$\mathcal{S}^d(G)$ is represented analogously by the **d-parallel matrix** $P^d(G)$

Combinatorial Rigidity in the Plane

Theorem 1 The following are equivalent:

(1) $G = (V, E)$ is 2-rigidity-independent, i.e., $\mathcal{R}^2(G) = 2^E$.

(2) (Recski's condition) For each $e \in E$, adding a parallel edge $\tilde{e}$ produces a graph that decomposes into two forests.



(3) (Laman's condition) For $\emptyset \neq F \subset E$,

$$|F| \leq 2|V(F)| - 3.$$

(Idea: edges are not concentrated in any one region of G. K_4 is the smallest simple counterexample.)

(4) $T_G(q, q)$ is monic of degree $r(G)$.

Problem: Generalize these criteria to arbitrary d.

Pictures, Planar Duality and Matroids

Picture of G: an arrangement of points and lines that correspond to vertices and edges of G

Picture space of G: the algebraic variety $X = X^d(G)$ of all pictures

Theorem 2 The following are equivalent:

(1) G is d-parallel independent;

(2) The d-dimensional picture space of G is irreducible;

(3) $T_G(q, q^{d-1})$ is monic of degree $r(G)$.

Corollary 3 (Planar Duality) $\mathcal{R}^2(G) = \mathcal{S}^2(G)$.

Corollary 4 The rigidity properties of G depend only on its underlying **graphic matroid**.

Rigidity Matroids of Matroids??

Motivated by Corollary 4...

...let's try to develop a version of rigidity theory in which the underlying objects of study are **matroids** rather than **graphs**.

Why do we want to do this?

- Provide combinatorial proofs of Laman's Theorem, Planar Duality Theorem, and other fundamental results of rigidity theory
- Generalize these theorems to a wider setting
- Explain geometric invariants (cross-ratio, tree polynomials) combinatorially
- Add to the toolbox of graph rigidity theory...
- ...and the theory of matroids themselves.

A Trinity of Independence Complexes

- There are **three** plausible notions of "d-rigidity-independence" for an arbitrary matroid M (with ground set E):

Combinatorial: M is ***d*-Laman-independent** if

$$d \cdot r(F) > |F| \qquad \text{for all } \emptyset \neq F \subset E$$

...provided that this condition gives a matroid (for which d)?

Linear algebraic: M is ***d*-rigidity-independent** if the rows of R are linearly independent

...where $R = R^d(M)$ is the rigidity matrix of M (generalizing the construction for the graphic case)

Geometric: M is ***d*-slope-independent** if $X^d(M)$ is irreducible

...where $X^d(M)$ is some matroidal analogue of the picture space

d-Laman Independence

Let $d \in (1,\infty)_{\mathbb{R}}$. The ***d*-Laman complex of *M*** is defined as

$$\mathcal{L}^d(M) \;=\; \big\{F \subset E : \; d \cdot r(F') > |F| \text{ for all } \emptyset \neq F' \subseteq F\big\}.$$

Theorem 5 $\;d \in \mathbb{Z} \quad \Longleftrightarrow \quad \mathcal{L}^d(M)$ is a matroid for every M.

Theorem 6 The following are equivalent:

(1) M is d-Laman-independent, i.e., $\mathcal{L}^d(M) = 2^E$.

(2) $T_M(q^{d-1}, q)$ is monic in q of degree $(d-1)r(M)$.

(3) M has an *Edmonds decomposition* as a disjoint union

$$E \;=\; I_1 \,\dot\cup\, I_2 \,\dot\cup\, \cdots \,\dot\cup\, I_d$$

where

- each I_k is independent in M; and
- there is no collection of nonempty subsets $J_1 \subset I_1, \dots, J_d \subset I_d$ such that $\overline{J_1} = \cdots = \overline{J_d}$.

(The proof relies on Edmonds' theorem on matroid partitioning.)

d-Slope Independence

Let M be represented by vectors $E = \{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$. For $0 < k < d \in \mathbb{N}$, let $\mathbb{G}(k, \mathbb{F}^d) = \{k\text{-dimensional subspaces of } \mathbb{F}^d\}$.

The **(k, d)-photo space** $X = X_{k,d}(M)$ is defined as

$$\left\{(\phi, W_1, \ldots, W_n) \in \mathrm{Hom}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}(k, \mathbb{F}^d)^n \ : \ \phi(v_i) \in W_i \quad (\forall i)\right\}.$$

(k, d)-slope independence: the map $X \to \mathbb{G}(k, \mathbb{F}^d)^n$ is dense.

(k, d)-slope complex of M:

$$\mathcal{S}^{k,d}(M) = \{A \subset E : \ M|_A \text{ is } (k, d)\text{-slope independent}\}.$$

Theorem 7 Let $m = \frac{d}{d-k}$. The following are equivalent:

(1) M is (k, d)-slope independent.

(2) The photo space X is an irreducible variety.

(3) M is m-Laman independent. (So $\mathcal{S}^{k,d}(M) = \mathcal{L}^m(M)$.)

Theorem 8 If $\mathbb{F}$ is the finite field $\mathbb{F}_q$, then $|X|$ is given by a certain Tutte polynomial specialization (involving q-binomial coefficients).

$\boldsymbol{d}$-Rigidity Independence

Let M be represented by vectors $E = \{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$. Let $\psi = (\psi_{ij})$ be a $(d \times r)$ matrix of transcendentals (regarded as a "generic" linear map $\mathbb{F}^r \to \mathbb{F}^d$).

Defn: The ***d*-rigidity matroid $\boldsymbol{\mathcal{R}^d(M)}$** is represented over $\mathbb{F}(\psi)$ by the vectors

$$\{v_i \otimes \psi(v_i) : \ i \in [n]\}$$

in $\mathbb{F}^r \otimes \mathbb{F}(\psi)^d$. (This generalizes the construction of $\mathcal{R}^d(G)$.)

Theorem 9 (The Nesting Theorem) Let M be a representable matroid and $d > 1$ an integer. Then:

$$\mathcal{S}^{1,d}(M) \ \subseteq \ \mathcal{R}^d(M) \ \subseteq \ \mathcal{L}^d(M) \ \subseteq \ \mathcal{S}^{d-1,d}(M).$$

Corollary 10 Equality holds throughout when $d = 2$.

(This generalizes both Laman's Theorem and the Planar Duality Theorem.)

Uniform Matroids

Let $|E| = n$. The **uniform matroid** $U_{r,n}$ is defined as

$$\{S \subset E \ : \ |S| \leq r\}.$$

- Every $U_{r,n}$ is representable over a suitable field (e.g., $\mathbb{R}$).
- $\mathcal{L}^d(U_{r,n})$ and $\mathcal{S}^{k,d}(U_{r,n})$ are uniform matroids for all k, d.

Example 1: $U_{2,3}$ (= graphic matroid of 3-cycle)

$$\mathcal{L}^d(U_{2,3}) = \begin{cases} U_{2,3} & \text{if } 1 < d \leq \frac{3}{2} \\ U_{3,3} & \text{if } d > \frac{3}{2} \end{cases}$$

$$\mathcal{S}^{1,d}(U_{2,3}) = \begin{cases} U_{3,3} & \text{if } d = 2 \\ U_{2,3} & \text{if } d = 3, 4, \dots \end{cases}$$

- For $\phi : \mathbb{F}^2 \to \mathbb{F}^2$, the slopes of the $\phi(v_i)$ may be specified freely
- For $\phi : \mathbb{F}^2 \to \mathbb{F}^d$ $(d > 2)$, the three lines $\phi(v_i)$ must be coplanar

$$\mathcal{R}^d(U_{2,3}) = \begin{cases} U_{3,3} & \text{if } d = 1 \\ U_{2,3} & \text{if } d = 2, 3, \dots \end{cases}$$

- Two sides of a triangle determine the third iff the triangle is flat!

Uniform Matroids (II)

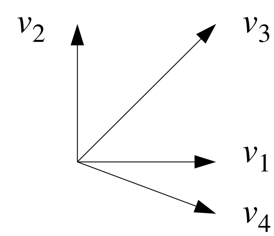
Example 2: $U_{2,4}$, represented as follows. (All representations are projectively equivalent to this one, up to the choice of μ.)

$v_1 = (1, 0)$

$v_2 = (0, 1)$

$v_3 = (1, 1)$

$v_4 = (1, \mu)$



$$\mathcal{L}^d(U_{2,4}) = \begin{cases} U_{2,4} & \text{if } 1 \leq d \leq \frac{3}{2} \\ U_{3,4} & \text{if } \frac{3}{2} < d \leq 2 \\ U_{4,4} & \text{if } d > 2 \end{cases}$$

$$\mathcal{S}^{1,d}(U_{2,4}) = \begin{cases} U_{3,4} & \text{if } d = 2 \\ U_{2,4} & \text{if } d = 3, 4, \ldots \end{cases}$$

- For $d > 1$, each $\phi : \mathbb{F}^2 \to \mathbb{F}^d$ preserves the cross-ratio μ, so there is an additional constraint on the slopes of the $\phi(v_i)$. Therefore

$$\mathcal{R}^d(U_{2,4}) = \begin{cases} U_{2,4} & \text{if } d = 1 \\ U_{3,4} & \text{if } d = 2, 3, \ldots. \end{cases}$$

Open Questions

1. Is $\mathcal{R}^d(M)$ a combinatorial invariant of M? That is, is it independent of the choice of representation of M, or at least of the ground field $\mathbb{F}$? Is the question easier if M is required to be graphic?

2. Give a combinatorial explanation for the identity

$$q^{d \cdot r(M)} |X_{d-k,d}(M^{\perp})| \;=\; q^{(d-k)n} |X_{k,d}(M)|$$

where r is the rank of M and $M^{\perp}$ is the dual matroid.

3. Describe the defining equations of the photo space. (These polynomials may be generating functions for certain bases of M.) What geometric invariants (such as the cross ratio) show up?

4. Study the singular locus of the photo space. (It is smooth iff M contains only loops and coloops.)

5. Explain the "dimension scaling phenomenon"

$$\mathcal{S}^{k,d}(M) = \mathcal{S}^{\lambda k, \lambda d}(M).$$

6. Generalize other rigidity-theoretic facts to the setting of matroids: for example, Henneberg's and Crapo's constructions of $\mathcal{L}^2$.