What Else Can You Count If You Can Count Trees?

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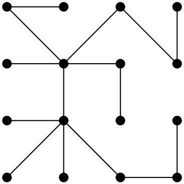
University of Kansas

Washburn University
March 3, 2020

Trees

Tree: a nonempty set of vertices connected by edges, so that

- there is a path between any two vertices (*connectedness*);
- there are no closed loops (*acyclicity*).



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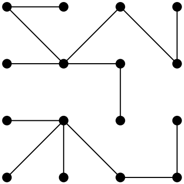
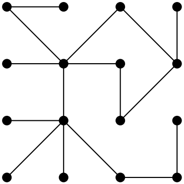


Not connected



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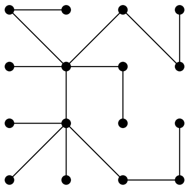

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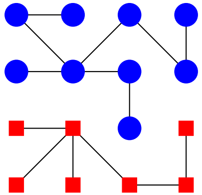

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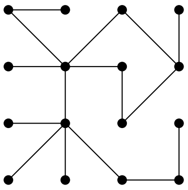

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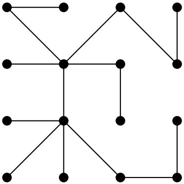

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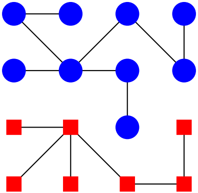

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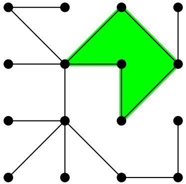

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Properties of Trees

1. Every tree with n vertices has **exactly $n-1$ edges**. (Any fewer and it cannot be connected; any more and it must contain a cycle.)

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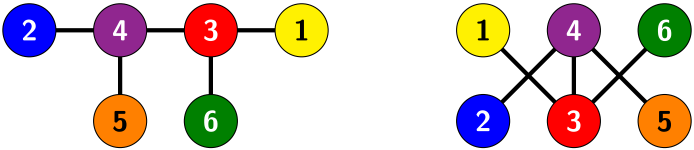
3. We only care about **which vertices are connected**, not how the tree is depicted on the page. These trees are the **same:**



Counting Trees

Question: How many different trees are there on n vertices?

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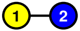
$n = 2$ 

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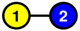

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3 trees

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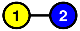

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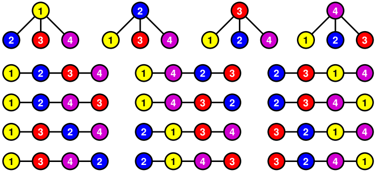
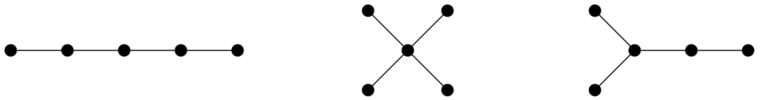
$n = 4$



16 trees

Counting Trees

For $n = 5$, there are three tree shapes:



$5!/2 = 60$ trees

5 trees

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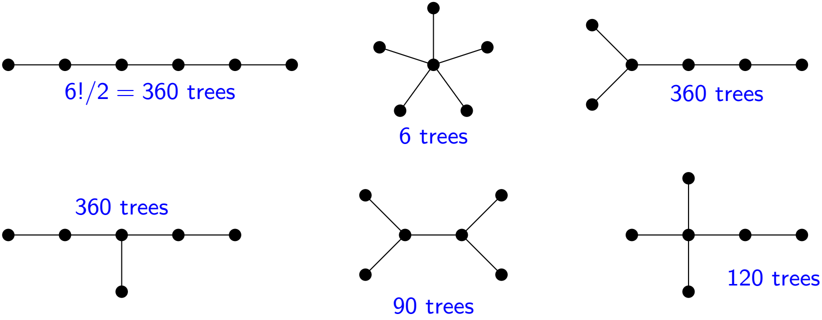
Total: **125** trees on 5 labeled vertices.

Counting Trees

For $n = 6$, there are six tree shapes:



Total: **1296** trees on 6 labeled vertices.

Counting Trees

Let $\tau(n)$ = number of labeled trees on n vertices.

n	$\tau(n)$
1	1
2	1
3	3
4	16
5	125
6	1296
7	16807
8	262144

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Theorem 1 ("Cayley's formula") For every integer n,

$$\boxed{\tau(n) = n^{n-2}.}$$

Yeah, But How Do You Prove That? (#1)

Given a tree T, we construct its **Prüfer code** $P(T)$ as follows.

- Find the **leaf** with the **smallest** label.
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- Repeat until just two vertices are left.

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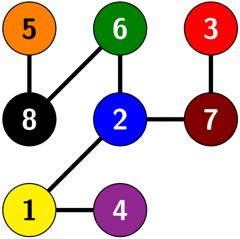

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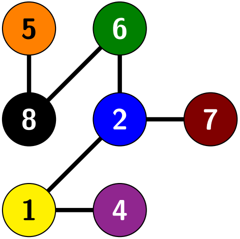

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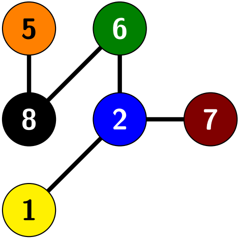

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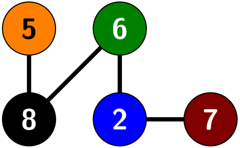

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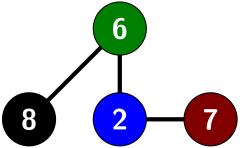

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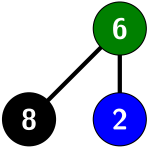

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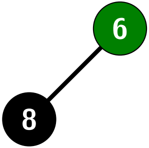

$$P(T) = (7, 1, 2, 8, 2, 6)$$

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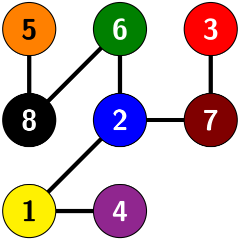

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Theorem 2: Every tree can be reconstructed from its Prüfer code. Therefore, there is a **bijection** (a one-to-one, onto function)

$$\{\text{trees on } n \text{ vertices}\} \xrightarrow{P} \{(p_1, \dots, p_{n-2}) : \ 1 \leq p_i \leq n\}$$

and the size of the right-hand set is clearly n^{n-2}.

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Theorem 3: The number of trees in which vertex i has exactly d_i neighbors is the coefficient of the monomial

$$x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

in the expansion of $x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$.

Yeah, But How Do You Prove That? (#2)

The **Matrix-Tree Theorem** (which dates back to 1845!) says that trees can be counted using linear algebra.

TL;DR: $\tau(n)$ is the determinant of the $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}$$

and you can work out for yourself that its eigenvalues are n (with multiplicity $n-2$) and 1.

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- Did I mention the pit full of snakes?

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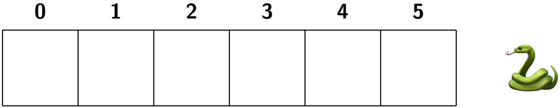

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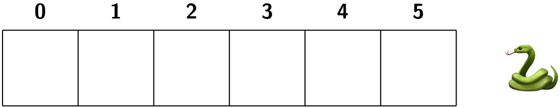

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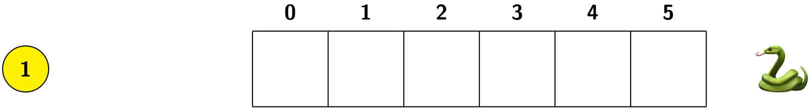

$p_1 = 0$

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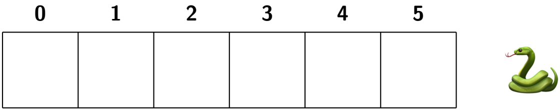

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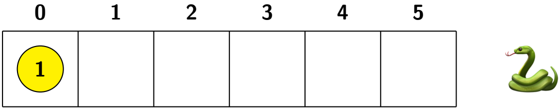

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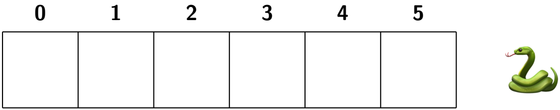

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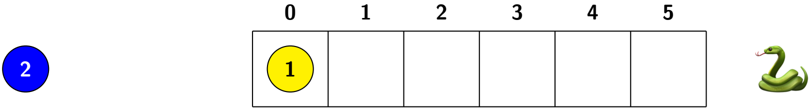
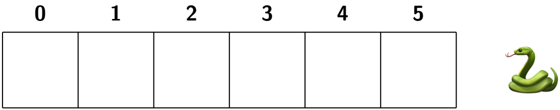


$p_2 = 3$

Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$



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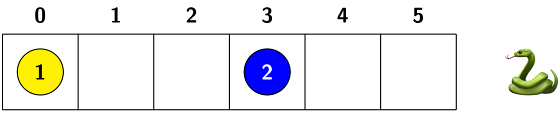

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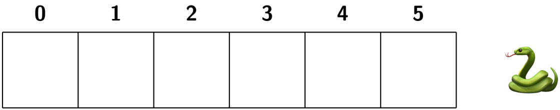

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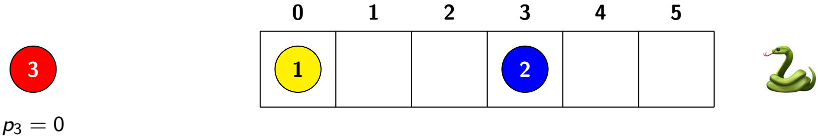

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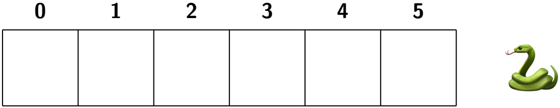

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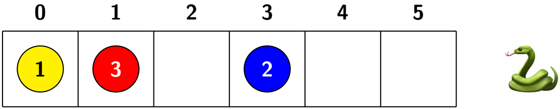

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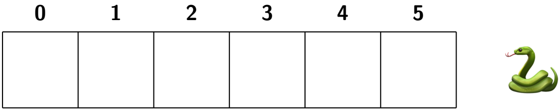

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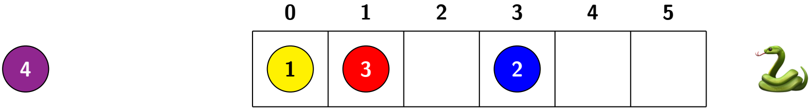

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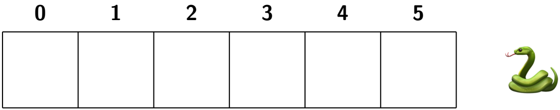

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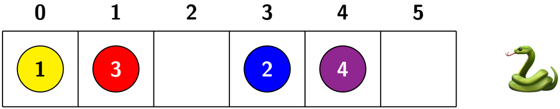

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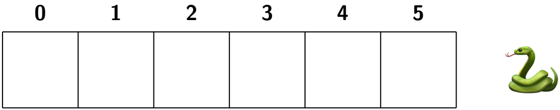

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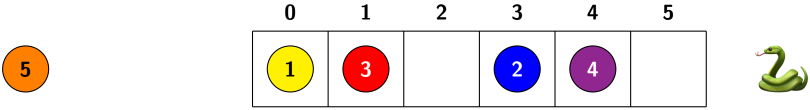
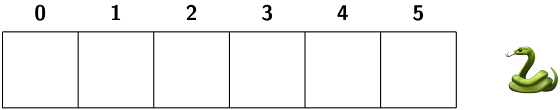


$p_5 = 3$

Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$



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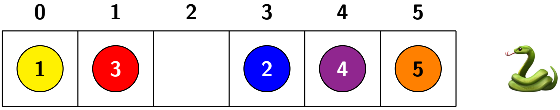

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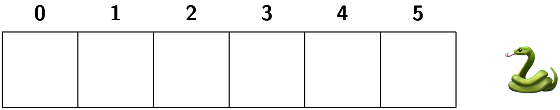

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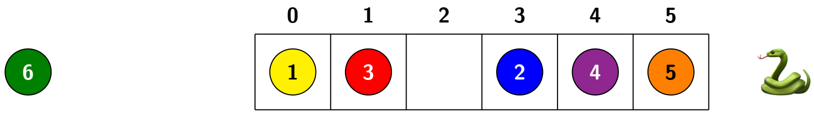
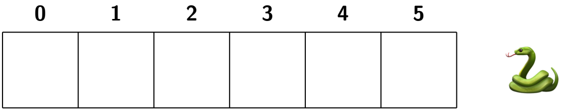


$p_6 = 0$

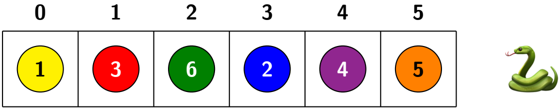
Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$



Parking Functions

Example #1 $(p_1, \ldots, p_6) = (0, 3, 0, 4, 3, 0)$

Success!



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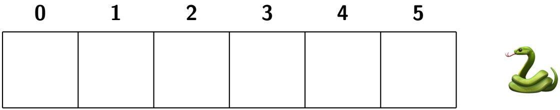

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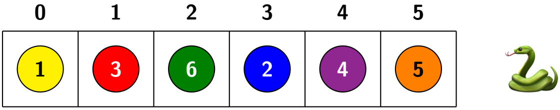
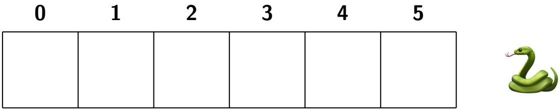
Success!



Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

$p_1 = 0$



Parking Functions

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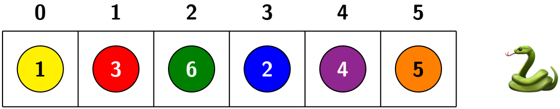

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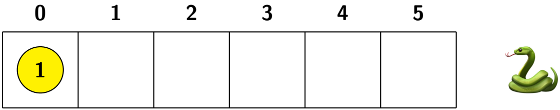

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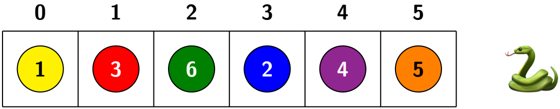
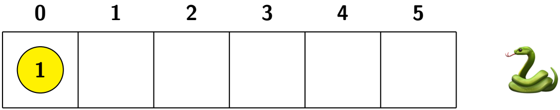


Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

$p_2 = 3$



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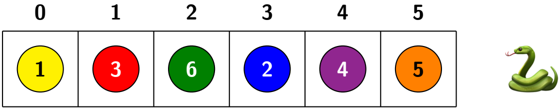

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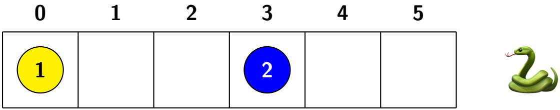


Parking Functions

Example #1 $(p_1, \ldots, p_6) = (0, 3, 0, 4, 3, 0)$

Success!

0	1	2	3	4	5
1	3	6	2	4	5

Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

3

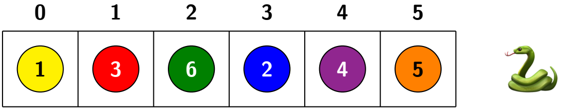
$p_3 = 3$

0	1	2	3	4	5
1			2		

Parking Functions

Example #1 $(p_1, \ldots, p_6) = (0, 3, 0, 4, 3, 0)$

Success!



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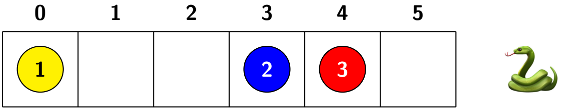

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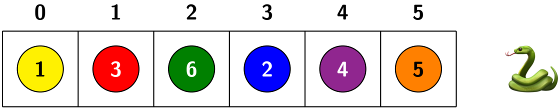
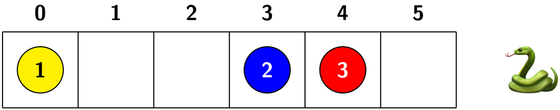


Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

$p_4 = 4$



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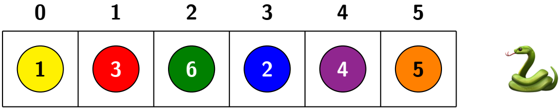
Success!



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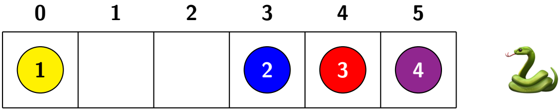


Parking Functions

Example #1 $(p_1, \ldots, p_6) = (0, 3, 0, 4, 3, 0)$

Success!

0	1	2	3	4	5
1	3	6	2	4	5

Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

5

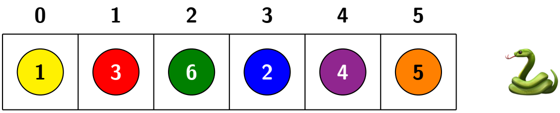
$p_5 = 3$

0	1	2	3	4	5
1			2	3	4

Parking Functions

Example #1 $(p_1, \ldots, p_6) = (0, 3, 0, 4, 3, 0)$

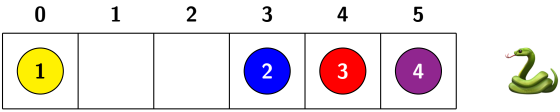
Success!



Example #2 $(p_1, \ldots, p_6) = (0, 3, 3, 4, 3, 0)$

$p_5 = 3$

Oops.



Parking Functions

Definition A sequence $\mathbf{p} = (p_1, \dots, p_n)$ is a **parking function (PF)** if it enables all cars to park without being eaten by snakes.

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(So any shuffle of a parking function is also a parking function.)

Theorem 5
There are $(n+1)^{n-1}$ parking functions of length n.

Parking Functions

$n = 1$:	0				
$n = 2$:	00	01			
		10			
$n = 3$:	000	001	011	002	012 021
		010	101	020	102 120
		100	110	200	201 210

$n = 4$ (up to shuffling):

0000				
0001	0002	0003		
0011	0012	0013	0022	0023
0111	0112	0113	0122	0123

Number of PFs up to shuffling: Catalan number $\frac{1}{n+1}\binom{2n}{n}$

A Rather Slick Way To Count Parking Functions

- Remove the snakepit. Replace it with an extra parking spot ($\#n$) and a return ramp (like an airport terminal).
- Number of preference lists $\mathbf{p}$ is now $(n+1)^n$.
- All cars will be able to park, and one spot $o(\mathbf{p})$ will be left open.
- Cyclically rotating $\mathbf{p}$ also rotates $o(\mathbf{p})$.
- Therefore, all spots are equally likely to be open.
- $\mathbf{p}$ is a parking function $\iff o(\mathbf{p}) = n$.
- Number of parking functions $= (n+1)^n/(n+1) = (n+1)^{n-1}$.

The Sandpile Model

Start with a bucket of sand. Make m piles.[1] Let s_i be the number of grains of sand in the i^{th} pile. Watch what happens.

- When a sandpile gets too big, it **topples**.

  Specifically, if $s_i \geq m$, then pile i spews sand in all directions, giving one grain of sand to each other pile and putting one grain back in the bucket.

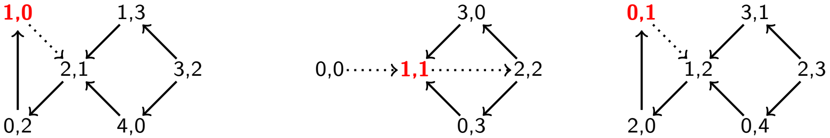
- If no pile is too big, add one grain from the bucket to each pile (**replenishment**)

How does the system evolve?

[1] I know a three-year-old who can help with this.

The Sandpile Model

Let $m = 2$. Record the state the model is in by the pair (s_1, s_2). (Dotted lines indicate replenishment steps.)



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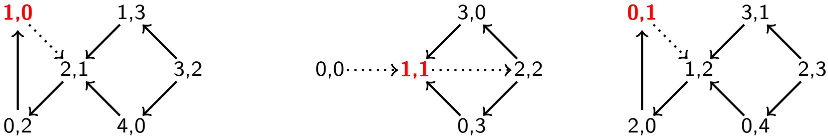

The states $(0, 1)$, $(1, 0)$, and $(1, 1)$ are called **critical**:

- no pile other than the sink can topple ("stability")
- these states appear repeatedly as the model evolves ("recurrence")

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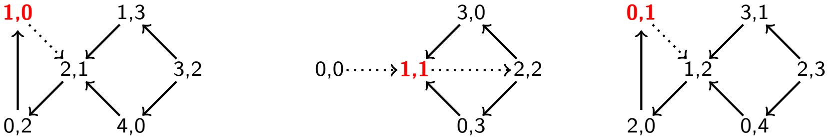


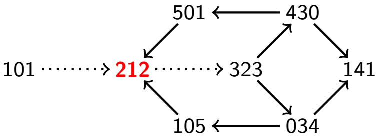
The states $(0,1)$, $(1,0)$, and $(1,1)$ are called **critical**:

- no pile other than the sink can topple ("stability")
- these states appear repeatedly as the model evolves ("recurrence")

Fact: Every initial state evolves to exactly one critical state.

The Sandpile Model

A possible evolution pattern for $m = 3$:



Complete list of critical states for $m = 3$:

> 222, 221, 212, 122, 211, 121, 112, 220, 202, 022,
> 012, 021, 102, 120, 201, 210.

Sandpiles and Shopping Sprees

Sandpile model (statistical physics)	**Dollar game** (economics)
Sandpiles	Consumers
Sand grains	Dollars
Big enough	Rich enough
Toppling	Shopping spree
Bucket	Bank
Replenishment	Economic stimulus package

Simultaneous Shopping Sprees

Suppose that Ani and Bob go on shopping sprees at the same time.

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Ani gives a dollar to Bob
Ani gives a dollar to Chris

Simultaneous Shopping Sprees

Suppose that Ani and Bob go on shopping sprees at the same time.

Ani gives a dollar to Bob	Bob gives a dollar to Ani
Ani gives a dollar to Chris	Bob gives a dollar to Chris

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Suppose that Ani and Bob go on shopping sprees at the same time.

~~Ani gives a dollar to Bob~~	~~Bob gives a dollar to Ani~~
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Simultaneous Shopping Sprees

Suppose that Ani and Bob go on shopping sprees at the same time.

~~Ani gives a dollar to Bob~~	~~Bob gives a dollar to Ani~~
Ani gives a dollar to Chris	Bob gives a dollar to Chris

Ani and Bob only need \$1 each to go on a **joint** shopping spree.

Simultaneous Shopping Sprees

Suppose that Ani and Bob go on shopping sprees at the same time.

~~Ani gives a dollar to Bob~~	~~Bob gives a dollar to Ani~~
Ani gives a dollar to Chris	Bob gives a dollar to Chris

Ani and Bob only need \$1 each to go on a **joint** shopping spree.

In a **joint shopping spree**, each consumer in a set X (not including the bank) gives \$1 to each consumer not in X (including the bank). This is possible if

$$s_i > m - |X| \qquad \forall x \in X.$$

Superstable States

A state of the dollar game is called **superstable** if no simultaneous shopping sprees are possible.

Theorem 2
$(s_1, \ldots, s_m)$ superstable $\iff$ $(m - s_1, \ldots, m - s_m)$ critical

Theorem 3
There is a bijection

$$\{\text{superstable states}\} \rightarrow \{\text{labeled trees on } n \text{ vertices}\}.$$

The proof uses the **Burning Algorithm** [Dhar, 1990].

Dhar's Burning Algorithm (A Sketch)

Let $n = m + 1$. Start with a superstable state $\mathbf{s} = (s_1, \ldots, s_{n-1})$.

- For each $i = 1, \ldots, n-1$, place s_i firefighters at vertex i.
- Set vertex n on fire.
- The fire tries to spread from burned vertices to unburned vertices. Unburned vertices can deploy firefighters to protect themselves. (A firefighter cannot be moved once deployed.)
- Superstability of $\mathbf{s}$ is precisely equivalent to the condition that the fire eventually reaches every vertex!
- The route that the fire takes is a tree!
- Algorithm is reversible: $\mathbf{s}$ can be reconstructed from the output tree!

Handicap Scoring

- Competitors in an individual event (e.g., marathon, bowling, pentathlon, Rubik's Cube) are seeded $1, 2, \ldots, n$.
  Lower numbered seed = stronger player.
- Each competitor i achieves a score $x_i \in \mathbb{R}$ (the higher the better).
- We want to level the playing field by comparing each pair of players head-to-head. For each $1 \leq i < j \leq n$:
  - If $x_i < x_j$ ("upset"), then the underdog j scores a point.
  - If $x_j < x_i < x_j + 1$ ("chalk"), then no one scores a point.
  - If $x_j + 1 < x_i$ ("blowout"), then the favorite i scores a point.

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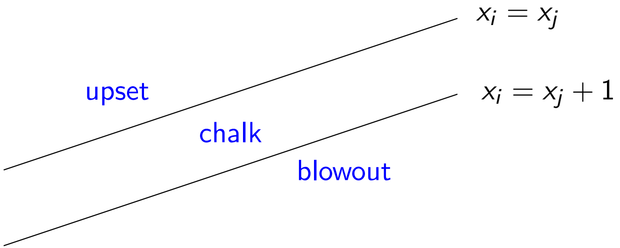


The Shi Arrangement

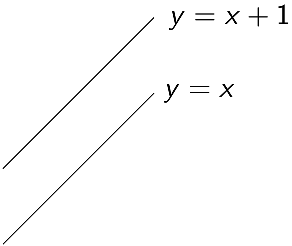
In order to understand the possible score vectors, we want to look at the hyperplanes in $\mathbb{R}^n$ defined by the equations

$$\begin{array}{llllll} x_1 = x_2, & x_1 = x_3, & \ldots, & x_i = x_j, & \ldots, & x_{n-1} = x_n, \\ x_1 = x_2 + 1, & x_1 = x_3 + 1, & \ldots, & x_i = x_j + 1, & \ldots, & x_{n-1} = x_n + 1. \end{array}$$

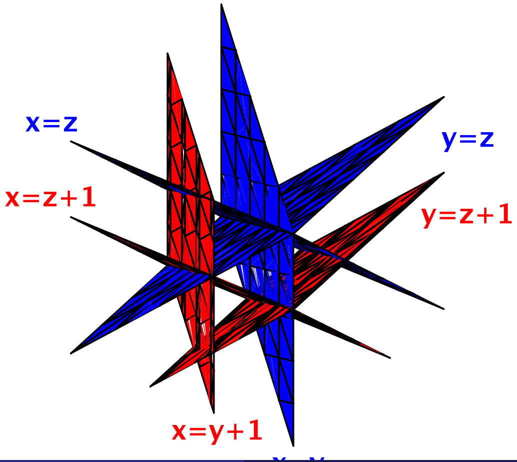
The **Shi arrangement** $Shi(n)$ is the set of all such hyperplanes.

The Shi arrangement separates $\mathbb{R}^n$ into regions that record the possible outcomes from this scoring system.

The Arrangement *Shi*(2)



The Arrangement *Shi*(3)



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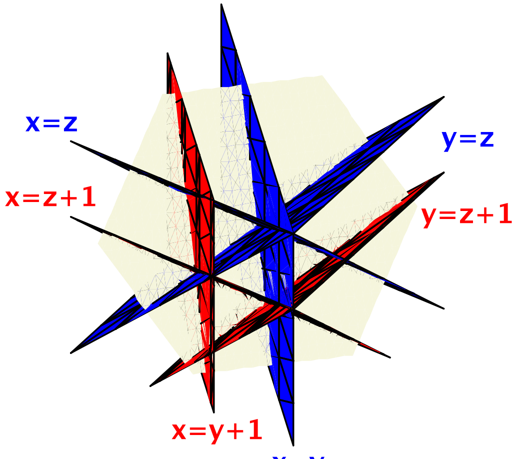
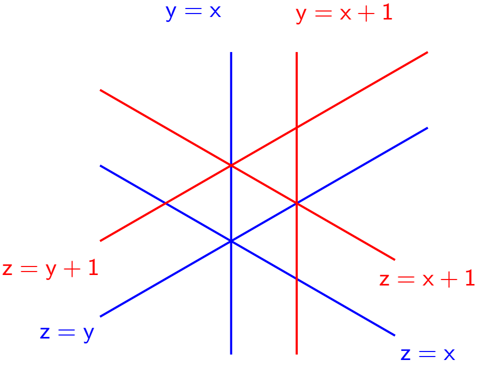
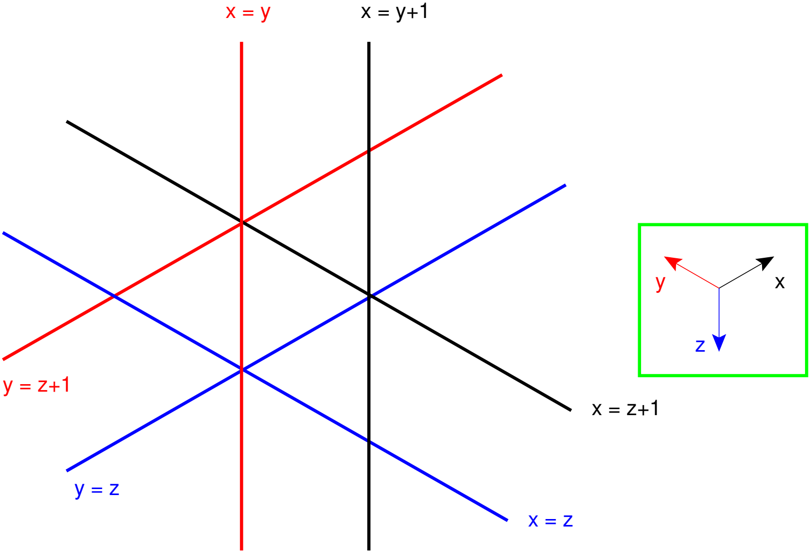
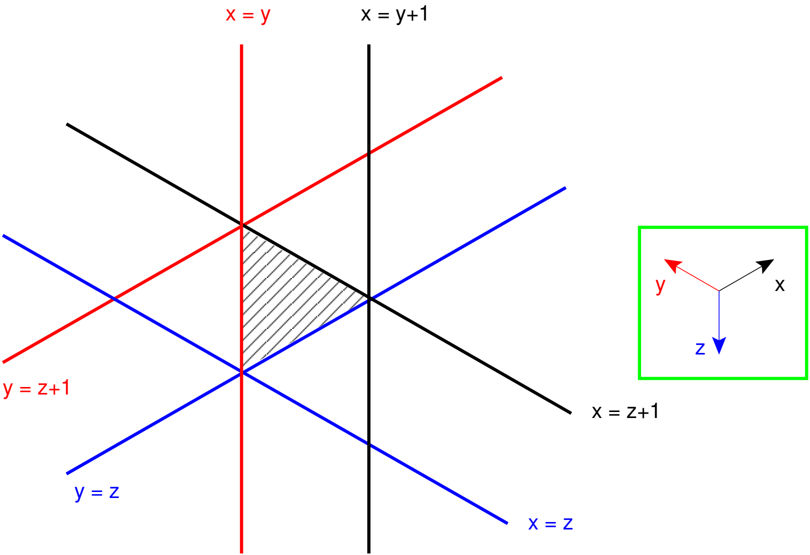



$y = x$
$y = x + 1$
$z = y + 1$
$z = x + 1$
$z = y$
$z = x$

Score Vectors for *Shi*(3)



Score Vectors for $Shi(3)$



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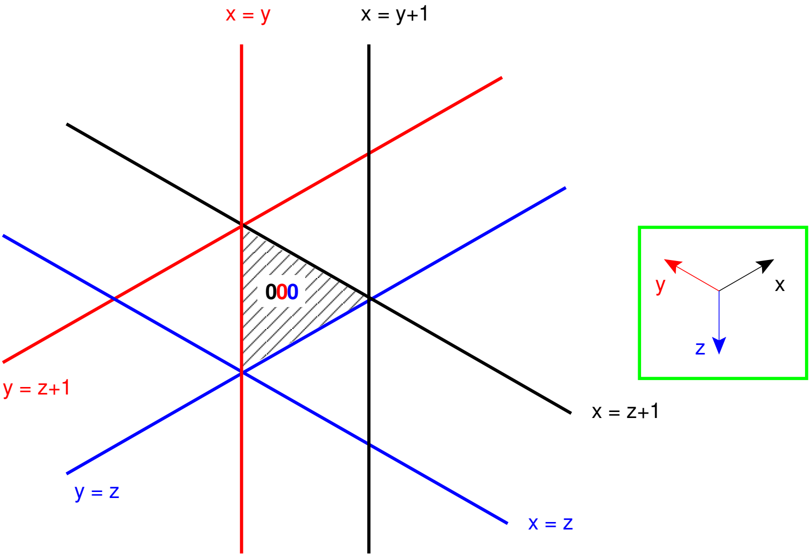

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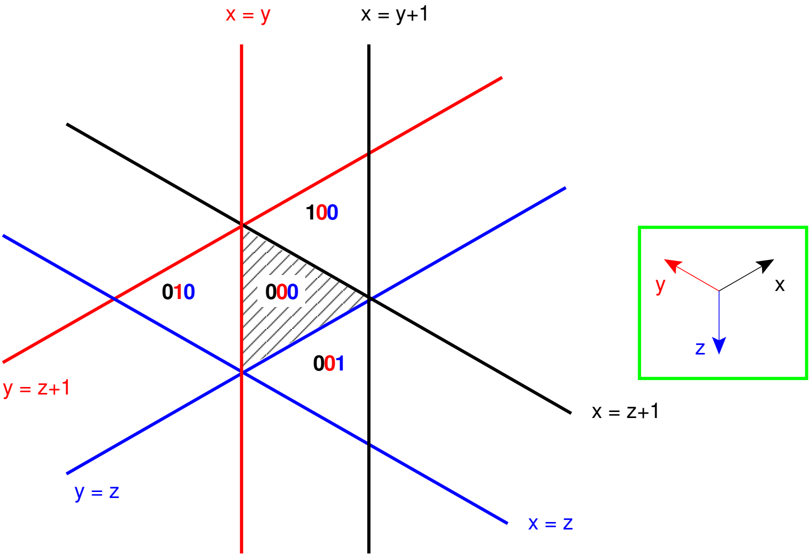

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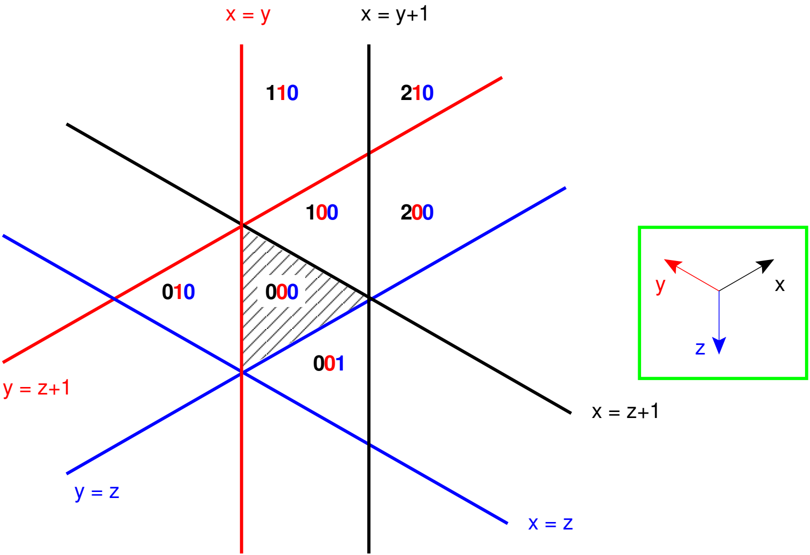

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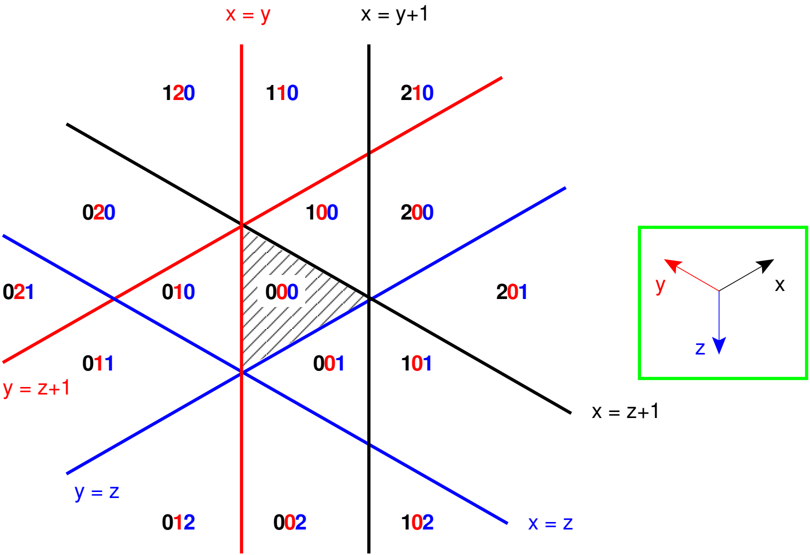

The Shi Arrangement

Theorem 6 [Pak and Stanley]
Labeling with score vectors gives a function

$$\{\text{regions of } Shi(n)\} \;\to\; \{\text{parking functions of length } n\}$$

that is a bijection!

In particular, the number of regions in $Shi(n)$ is $(n+1)^{n-1}$.

Conclusion

The numbers $(n+1)^{n-1}$ count lots of things:

- labeled trees on $n+1$ vertices,
- long-term behaviors of the sandpile model with n vertices plus a sink,
- superstable states of the dollar game with n vertices plus a bank,
- parking functions for n cars,
- regions of the Shi arrangement in $\mathbb{R}^n$,
- and, quite possibly, other beautiful combinatorial structures that you will discover yourself (and please tell me).

Thank you!