Simplicial and Cellular Spanning Trees, II: Applications

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Shifted Complexes

Definition A simplicial complex Δ on vertices $[n]$ is **shifted** if for all $F \in \Delta$, $i \in \Delta$, $j \notin \Delta$, and $j < i$, we have $F \setminus \{i\} \cup \{j\} \in \Delta$.

Example If Δ is shifted and $235 \in \Delta$, then Δ must also contain the faces 234, 135, 134, 125, 124, 123.

- Shifted complexes of dimension 1 are *threshold graphs*.

Shifted Complexes

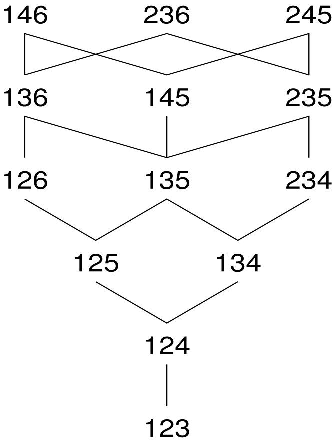
Define the **componentwise (partial) order** on $(d+1)$-sets of positive integers

$$A = \{a_1 < a_2 < \cdots < a_{d+1}\},$$
$$B = \{b_1 < b_2 < \cdots < b_{d+1}\}$$

by

$$A \preceq B \iff a_i \leq b_i \text{ for all } i.$$

- The set of facets of a shifted complex is a *lower order ideal* with respect to $\preceq$.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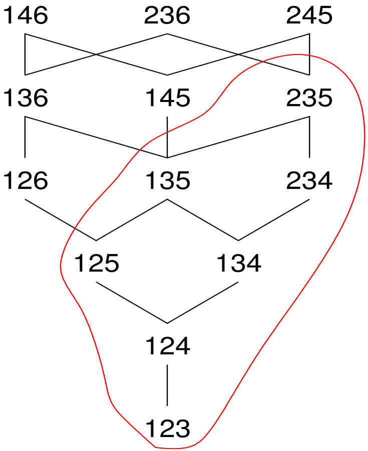
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Shifted Complexes

Proposition Shifted complexes are shellable, hence Cohen-Macaulay, hence metaconnected.

Theorem [Duval–Reiner 2001]
For Δ shifted, the eigenvalues of the unweighted Laplacian L are given by the transpose of the vertex/facet degree sequence.

- In particular, shifted complexes are Laplacian integral.

The Combinatorial Fine Weighting

Let Δ^d be a shifted complex on vertices $[n]$.
For each facet $A = \{a_1 < a_2 < \cdots < a_{d+1}\}$, define

$$x_A = \prod_{i=1}^{d+1} x_{i,a_i} \ .$$

Example If $\Upsilon = \langle 123, 124, 134, 135, 235 \rangle$ is a simplicial spanning tree of Δ, its contribution to h_2 is

$$x_{1,1}^4 x_{1,2} x_{2,2}^2 x_{2,3}^3 x_{3,3} x_{3,4}^2 x_{3,5}^2 \ .$$

The Algebraic Fine Weighting

For faces $A \subset B \in \Delta$ with $\dim A = i - 1$, $\dim B = i$, define

$$X_{AB} = \frac{\uparrow^{d-i} x_B}{\uparrow^{d-i+1} x_A}$$

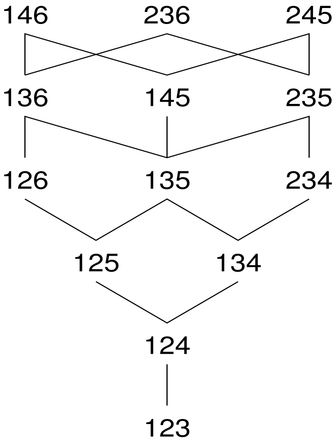
where $\uparrow x_{i,j} = x_{i+1,j}$.

- Weighted boundary maps $\boldsymbol{\partial}$ satisfy $\boldsymbol{\partial\partial} = 0$.
- Laplacian eigenvalues are the same as those for the combinatorial fine weighting, except for denominators.

Critical Pairs

Definition A **critical pair** of a shifted complex Δ^d is an ordered pair (A, B) of $(d+1)$-sets of integers, where

- $A \in \Delta$ and $B \notin \Delta$; and
- B covers A in componentwise order.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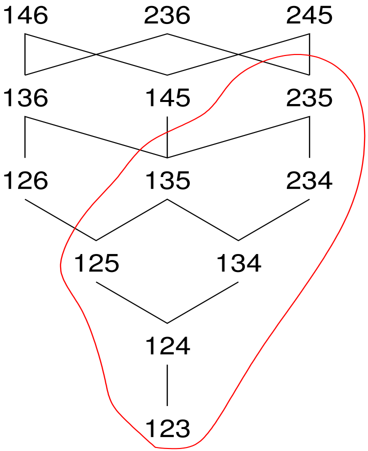
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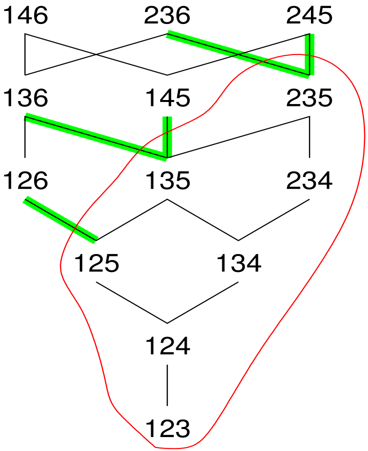
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The Signature of a Critical Pair

Let (A, B) be a critical pair of a complex Δ:

$$A = \{a_1 < a_2 < \cdots < a_i < \cdots < a_{d+1}\},$$

$$B = A \setminus \{a_i\} \cup \{a_i + 1\}.$$

Definition The **signature** of (A, B) is the ordered pair

$$\big(\{a_1, a_2, \ldots, a_{i-1}\},\ a_i\big).$$

Finely Weighted Laplacian Eigenvalues

Theorem [Duval–Klivans–JLM 2007]

Let Δ^d be a shifted complex.

Then the finely weighted Laplacian eigenvalues of Δ are specified completely by the signatures of critical pairs of Δ.

$$\text{signature } (S, a) \quad \Longrightarrow \quad \text{eigenvalue } \frac{1}{\uparrow X_S} \sum_{j=1}^{a} X_{S \cup j}$$

Examples of Finely Weighted Eigenvalues

- Critical pair (135,145); signature (1,3):

$$\frac{X_{11}X_{21} + X_{11}X_{22} + X_{11}X_{23}}{X_{21}}$$

- Critical pair (235,236); signature (23,5):

$$\frac{X_{11}X_{22}X_{33} + X_{12}X_{22}X_{33} + X_{12}X_{23}X_{33} + X_{12}X_{23}X_{34} + X_{12}X_{23}X_{35}}{X_{22}X_{33}}$$

Sketch of Proof

- Calculate eigenvalues of Δ in terms of eigenvalues of the *deletion* and *link*:

$$\begin{aligned} \operatorname{del}_1 \Delta &= \{F \in \Delta \mid 1 \notin F\}, \\ \operatorname{link}_1 \Delta &= \{F \in \Delta \mid 1 \notin F,\ F \cup \{1\} \in \Delta\}. \end{aligned}$$

  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.

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- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$

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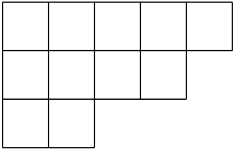
  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.
- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$
- "Here see ye two recurrences, and lo! they are the same."

Consequences of the Main Theorem

- Passing to the unweighted version (by setting $x_{i,j} = 1$ for all i, j) recovers the Duval–Reiner theorem.
- Special case $d = 1$: recovers known weighted spanning tree enumerators for threshold graphs (Remmel–Williamson 2002; JLM–Reiner 2003).
- A shifted complex is determined by its set of signatures, so we can "hear the shape of a shifted complex" from its Laplacian spectrum.

Ferrers Graphs

A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.



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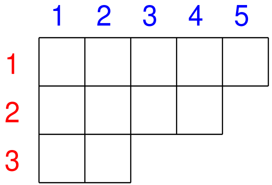

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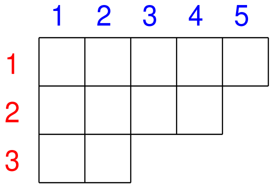

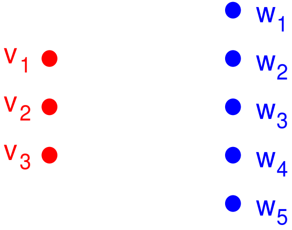

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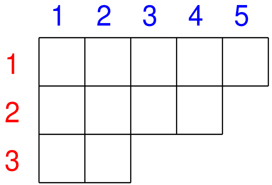

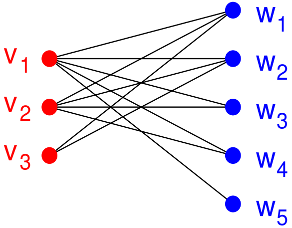

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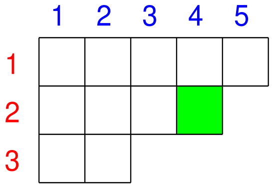

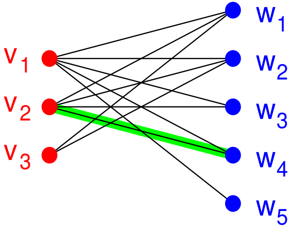

Ferrers Graphs

Ferrers graphs are bipartite analogues of threshold graphs.

- Degree-weighted spanning tree enumerator for Ferrers graphs: Ehrenborg and van Willigenburg (2004)
- Formula can also be derived from our finely weighted spanning tree enumerator for a threshold graph
- Higher-dimensional analogues?

Color-Shifted Complexes

Let Δ be a complex on $V = \bigcup_i V_i$, where

$$V_1 = \{v_{11}, \ldots, v_{1r_1}\}, \ \ldots, \ V_n = \{v_{n1}, \ldots, v_{nr_n}\}.$$

are disjoint vertex sets ("color classes").

Definition Δ is **color-shifted** if

- no face contains more than one vertex of the same color; and
- if $\{v_{1b_1}, \ldots, v_{nb_n}\} \in \Delta$ and $a_i \leq b_i$ for all i, then $\{v_{1a_1}, \ldots, v_{na_n}\} \in \Delta$.

Color-Shifted Complexes

- Color-shifted complexes generalize Ferrers graphs (Ehrenborg–van Willigenburg) and complete colorful complexes (Adin)
- Not in general Laplacian integral. . .
- . . . but they do seem to have nice degree-weighted spanning tree enumerators.

Matroid Complexes

Definition A pure simplicial complex Δ is a **matroid complex** if its facets form a matroid basis system:

- if F, G are facets and $i \in F \setminus G$,
- then there exists $j \in G \setminus F$ such that $F \setminus \{i\} \cup \{j\}$ is a facet.

Theorem [Kook–Reiner–Stanton 1999] Matroid complexes are Laplacian integral.

- Experimentally, degree-weighted spanning tree enumerators seem to have nice factorizations.