Simplicial and Cellular Spanning Trees, I: General Theory

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Graphs and Spanning Trees

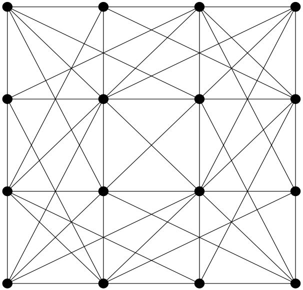
$G = (V, E)$: simple connected graph

Definition A **spanning tree of G** is a subgraph (V, T) such that

1. (V, T) is **connected** (every pair of vertices is joined by a path);
2. (V, T) is **acyclic** (contains no cycles);
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Graphs and Spanning Trees



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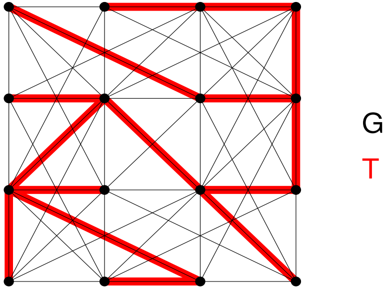
Graphs and Spanning Trees



Counting Spanning Trees

Let $\tau(G)$ denote the number of spanning trees of G.

Graph G	$\tau(G)$
Any tree	1
C_n (cycle on n vertices)	n
K_n (complete graph on n vertices)	n^{n-2} (Cayley)
$K_{p,q}$ (complete bipartite graph)	$p^{q-1}q^{p-1}$ (Fiedler-Sedláçek)
Q_n (n-dimensional hypercube)	$2^{2^n-k-1}\prod_{k=2}^{n} k^{\binom{n}{k}}$

The Laplacian Matrix

Let $G = (V, E)$ be a graph with $V = [n] = \{1, 2, \ldots, n\}$.

Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -1 & \text{if } i, j \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

- L is a real symmetric matrix
- $L = MM^{tr}$, where M is the signed incidence matrix of G

The Matrix-Tree Theorem

Matrix-Tree Theorem, Version I: Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$

Matrix-Tree Theorem, Version II: Let L_i be the *reduced Laplacian* obtained by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i.$$

[Kirchhoff, 1847]

The Matrix-Tree Theorem

Sketch of proof:

1. Expand $\det L_i$ using the Binet-Cauchy formula:

$$\det L_i = \det M_i\, M_i^{tr} = \sum_{\substack{T \subset E \\ |T|=n-1}} (\det M_T)^2$$

   where M_T = square submatrix of M_i with columns T

2. Show that

$$\det M_T = \begin{cases} \pm 1 & \text{if } T \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

Weighted Spanning Tree Enumerators

Idea: Let's record combinatorial information about a spanning tree T by assigning it a monomial weight x_T.

(e.g., vertex degrees; number of edges in specified sets; etc.)

Definition The **weighted spanning tree enumerator of G** is the generating function

$$\sum_{T \in \mathscr{T}(G)} x_T$$

where $\mathscr{T}(G)$ denotes the set of spanning trees of G.

Weighted Spanning Tree Enumerators

The weighted spanning tree enumerator of a graph

- reveals much more detailed combinatorial information about spanning trees of G than merely counting them
  - (particularly when it factors!)
- can suggest bijective proofs of formulas for $\tau(G)$

The Weighted Laplacian

Introduce an indeterminate e_{ij} for each pair of vertices i, j.
Set $e_{ij} = e_{ji}$, and if i, j are not adjacent, then set $e_{ij} = 0$.

The **weighted Laplacian of G** is the $n \times n$ matrix $\hat{L} = [\hat{\ell}_{ij}]$, where

$$\hat{\ell}_{ij} = \begin{cases} \sum_{j \neq i} e_{ij} & \text{if } i = j, \\ -e_{ij} & \text{if } i \neq j. \end{cases}$$

- Setting $e_{ij} = 1$ for each edge ij recovers the usual Laplacian L.

The Weighted Matrix-Tree Theorem

Weighted Matrix-Tree Theorem I: If $0, \hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{n-1}$ are the eigenvalues of $\hat{L}$, then

$$\sum_{T \in \mathscr{T}(G)} \prod_{ij \in T} e_{ij} = \frac{\hat{\lambda}_1 \hat{\lambda}_2 \cdots \hat{\lambda}_{n-1}}{n}.$$

Weighted Matrix-Tree Theorem II: If $\hat{L}_{k\ell}$ is obtained by deleting the k^{th} row and ℓ^{th} column of $\hat{L}$, then

$$\sum_{T \in \mathscr{T}(G)} \prod_{ij \in T} e_{ij} = (-1)^{k+\ell} \det \hat{L}_{k\ell}.$$

Example: The Cayley-Prüfer Theorem

Weight spanning trees of complete graph K_n by degree sequence:

$$x_T = \prod_{i=1}^{n} x_i^{\deg_T(i)}$$

Theorem [Cayley-Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_T = (x_1 x_2 \cdots x_n)(x_1 + x_2 + \cdots + x_n)^{n-2}.$$

(Setting $x_i = 1$ for all i recovers Cayley's formula.)

Example: The Cayley-Prüfer Theorem

Theorem [Cayley-Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_T = (x_1 x_2 \cdots x_n)(x_1 + x_2 + \cdots + x_n)^{n-2}.$$

Combinatorial proof: the *Prüfer code*, a bijection

$$P : \mathscr{T}(K_n) \to [n]^{n-2}$$

where $\deg_T(i) = 1 +$ number of i's in $P(T)$.

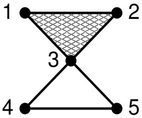
More Weighted Spanning Tree Enumerators

- $K_{p,q}$: degree sequence (bijection: Hartsfield–Werth)
- Threshold graphs: degree sequence and more (Remmel–Williamson)
- Ferrers graphs: degree sequence (Ehrenborg–van Willigenburg)
- Hypercubes: direction and facet degrees (JLM–Reiner; bijection??)

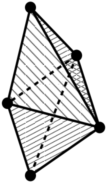
Simplicial Complexes

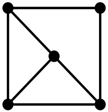
Definition A **simplicial complex on vertex set V** is a family Δ of subsets of V such that

- $\{v\} \in \Delta$ for every $v \in V$;
- If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.



{{}, 1, 2, 3, 4, 5,
12, 13, 23, 34, 35, 45,
123}

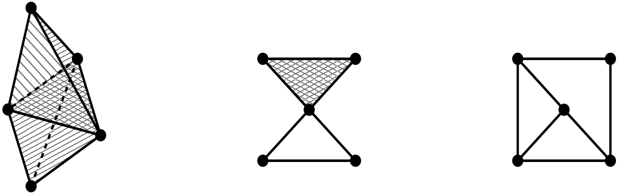




Simplicial Complexes

- Elements of Δ are called **faces**.
- Maximal faces are called **facets**.
- $\dim F = |F| - 1; \quad \dim \Delta = \max\{\dim F \mid F \in \Delta\}$.
- Δ is *pure* if all facets have equal dimension.
- $f_i(\Delta) =$ number of i-dimensional faces.
- The **k-skeleton** is $\Delta_{(k)} = \{F \in \Delta \mid \dim F \leq k\}$.
- A graph is just a simplicial complex of dimension 1.

Simplicial Complexes



pure; dim=2
$f(\Delta)=(5,9,7)$

not pure; dim=2
$f(\Delta)=(5,6,1)$

pure; dim=1
$f(\Delta)=(5,7)$

Simplicial Homology

R = commutative ring with identity (typically $\mathbb{Z}$ or $\mathbb{Q}$)
$C_i(\Delta)$ = free R-module on i-dimensional faces of Δ

Δ has natural **boundary** and **coboundary maps**

$$\partial_i : C_i \to C_{i-1}, \qquad \partial_i^* : C_{i-1} \to C_i$$

such that

$$\partial_i \circ \partial_{i+1} = \partial_{i+1}^* \circ \partial_i^* = 0.$$

Simplicial Homology

Definition The i^{th} **reduced simplicial homology group** of Δ is

$$\tilde{H}_i(\Delta; R) = \ker \partial_i \,/\, \operatorname{im} \partial_{i+1}.$$

- Homology groups over $\mathbb{Q}$ measure the holes in Δ.
- Homology groups over $\mathbb{Z}$ measure holes (the free part) and "twisting" (the torsion part).

Definition The i^{th} **Betti number of Δ** is

$$\tilde{\beta}_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}_i(\Delta, \mathbb{Q}).$$

Simplicial Homology

Let G be a graph (a 1-dimensional simplicial complex).

- $\tilde{\beta}_0(G) =$ (number of connected components of G) $- 1$
- $\tilde{\beta}_1(G) =$ number of edges that need to be deleted to make G acyclic

- ∂_1 is the signed vertex-edge incidence matrix M.
- The Laplacian of G is $L = \partial_1 \partial_1^*$.

Simplicial Spanning Trees

Let Δ^d be a simplicial complex (i.e., $\dim \Delta = d$).

Let $\Upsilon \subset \Delta$ be a subcomplex with $\Upsilon_{(d-1)} = \Delta_{(d-1)}$.

Definition Υ is a **simplicial spanning tree** (SST) of Δ if

1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$;
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$.

Simplicial Spanning Trees

Conditions for $\Upsilon \subset \Delta^d$ to be an SST:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$ ("connected");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- Any two of these conditions together imply the third
- When $d = 1$, coincides with the usual definition

Metaconnectedness

Denote by $\mathscr{T}(\Delta)$ the set of simplicial spanning trees of Δ.

Proposition $\mathscr{T}(\Delta) \neq \emptyset$ if and only if Δ has the homology type of a wedge of d-spheres:

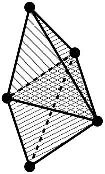
$$\tilde{\beta}_j(\Delta) = 0 \qquad \forall j < \dim \Delta.$$

Equivalently,

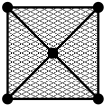
$$|\tilde{H}_j(\Delta; \mathbb{Z})| < \infty \qquad \forall j < \dim \Delta.$$

Such a complex is called **metaconnected**.

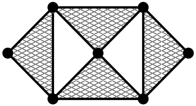
Metaconnectedness



metaconnected



metaconnected



not metaconnected

Metaconnectedness

- Every acyclic complex is metaconnected.
- Every Cohen-Macaulay complex is metaconnected (by Reisner's theorem), including:
  - 0-dimensional complexes
  - connected graphs
  - simplicial spheres
  - shifted complexes
  - matroid complexes
  - many other complexes arising in algebra and combinatorics

Examples of SSTs

Example If $\dim \Delta = 0$, then $\mathscr{T}(\Delta) = \{\text{vertices of } \Delta\}$.

Example If Δ is $\mathbb{Q}$-acyclic, then $\mathscr{T}(\Delta) = \{\Delta\}$.

- Includes complexes that are not $\mathbb{Z}$-acyclic, such as $\mathbb{RP}^2$.

Example If Δ is a simplicial sphere, then

$$\mathscr{T}(\Delta) = \{\Delta \setminus \{F\} \mid F \text{ a facet of } \Delta\}.$$

- Simplicial spheres are the analogues of cycle graphs.

Kalai's Theorem

Let Δ be the d-skeleton of the n-vertex simplex:

$$\Delta = \{F \subset [n] \mid \dim F \leq d\}.$$

Theorem [Kalai 1983]

$$\sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

- Reduces to Cayley's formula when $d = 1$ ($\Delta = K_n$).
- Adin (1992): Analogous formula for *complete colorful complexes* (generalizing Fiedler-Sédlaçek formula for $K_{p,q}$)

Simplicial Analogues of Graph Invariants

Let Δ^d be a metaconnected simplicial complex.

$$C_{i-1}(\Delta) \xleftarrow{\partial_i^*} C_i(\Delta) \xrightarrow{\partial_i} C_{i-1}(\Delta)$$

$L_i = \partial_i \partial_i^*$ (the **up-down Laplacian**)

s_i = product of nonzero eigenvalues of L_i

$$h_i = \sum_{\Upsilon \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(\Upsilon)|^2$$

The Simplicial Matrix-Tree Theorem — Version I

s_i = product of nonzero eigenvalues of Laplacian L_i

$$h_i = \sum_{\Upsilon \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(\Upsilon)|^2$$

Theorem [Duval–Klivans–JLM 2006]

$$h_d = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2.$$

Special Cases

- When Δ is a graph on n vertices, the theorem says that

$$h_1 \;=\; \frac{s_1}{h_0}|\tilde{H}_{-1}(\Delta)|^2 \;=\; \frac{s_1}{n}$$

  which is the classical Matrix-Tree Theorem.

- If $\tilde{H}_i(\Delta, \mathbb{Z}) = 0$ for $i \leq d-2$, then

$$h_d \;=\; \frac{s_d\, s_{d-2}\, \cdots}{s_{d-1}\, s_{d-3}\, \cdots}$$

The Simplicial Matrix-Tree Theorem — Version II

Δ^d = simplicial complex

$\Gamma \in \mathscr{T}(\Delta_{(d-1)})$

∂_Γ = restriction of ∂_d to faces not in Γ

$L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [Duval–Klivans–JLM 2006]

$$h_d = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma .$$

Simplicial Matrix-Tree Theorems

Theorem (SMTT–I: product of eigenvalues)

$$h_d = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2$$

Theorem (SMTT–II: reduced Laplacian)

$$h_d = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma$$

- Version II is more useful for computing h_d directly.
- In many cases, the $\tilde{H}_{d-2}$ terms are trivial.

Weighted SST Enumeration

- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply the F^{th} column of ∂ by x_F
- Weighted Laplacian $\mathbf{L} = \boldsymbol{\partial}\boldsymbol{\partial}^*$
- Weighted analogues of s_i and h_i :

$$\mathbf{s}_i = \text{product of nonzero eigenvalues of } \mathbf{L}_i$$

$$\mathbf{h}_i = \sum_{T \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(T)|^2 \prod_{F \in T} x_F^2$$

Weighted Simplicial Matrix-Tree Theorems

Weighted Simplicial Matrix-Tree Theorem I

$$\mathbf{h}_i = \frac{\mathbf{s}_i}{\mathbf{h}_{i-1}} \, |\tilde{H}_{i-2}(\Delta)|^2$$

Weighted Simplicial Matrix-Tree Theorem II

$$\mathbf{h}_i \; = \; \frac{|\tilde{H}_{i-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{i-2}(\Gamma;\mathbb{Z})|^2} \; \det \mathbf{L}_\Gamma$$

where $\Gamma \in \mathscr{T}(\Delta_{(i-1)})$

Weighted SST Enumeration

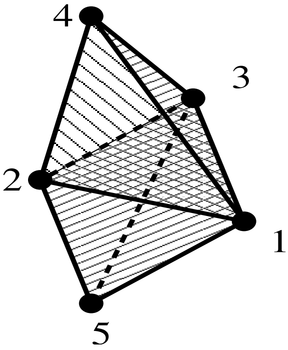
As in the graphic case, we can use weights to obtain finer enumerative information about simplicial spanning trees.

In order for the weighted simplicial spanning tree enumerators to factor, we need $\mathbf{L}$ to have integer eigenvalues.

That is, Δ must be **Laplacian integral**.

- Shifted complexes
- Matroid complexes
- Others?

Example: The Bipyramid With Equator



Vertices: 1, 2, 3, 4, 5

Edges: All but 45

Facets: 123, 124, 134, 234, 125, 135, 235

$f(\Delta) = (5, 9, 7)$

"Equator": the facet 123

Example: The Bipyramid With Equator

$$\begin{aligned}\Delta &= \text{bipyramid with equator}\\ &= \langle 123, 124, 134, 234, 125, 135, 235\rangle\end{aligned}$$

- For each facet $F = ijk$, set $x_F = x_i x_j x_k$.

Enumeration of SSTs of Δ by degree sequence:

$$\begin{aligned}\mathbf{h}_2 &= \sum_{T \in \mathscr{T}(\Delta)} \prod_{i \in V} x_i^{\deg_T(i)}\\ &= x_1^3 x_2^3 x_3^3 x_4^2 x_5^2 (x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)\end{aligned}$$

A represented matroid is a collection of vectors.

A matroid structure on a finite set says, "If this were a collection of vectors, then here is a list of which subsets would be linearly independent, bases, etc."

If X is a simplicial complex, then the simplicial matroid is the matroid represented by the columns of its boundary map.

Simplicial spanning trees are then exactly the bases of this matroid.

independence system is...

Cell complexes - argument that they are natural for combinatorialists to think about