On the Eigenvalues of Simplicial Rook Graphs

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AMS Southeastern Sectional Meeting
Tulane University
October 13–14, 2012

Simplicial Rook Graphs

Let $d, n \in \mathbb{N}$, and let $n\Delta^{d-1}$ denote the dilated simplex

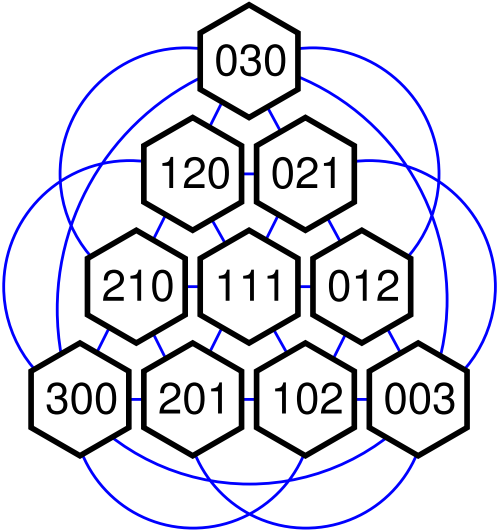
$$\{\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d \colon \sum_{i=1}^{d} v_i = n\}.$$

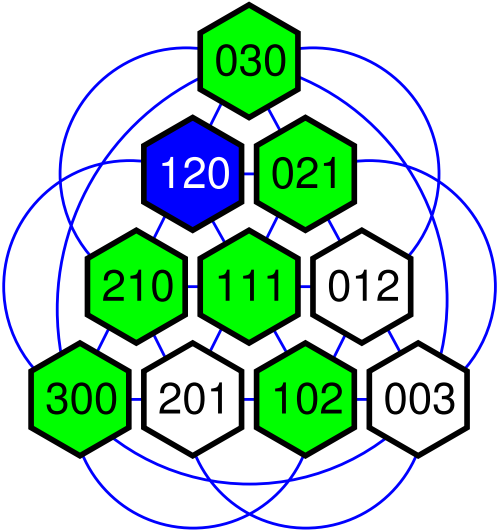
The simplicial rook graph $SR(d, n)$ is the graph with vertices

$$V(d, n) = n\Delta^{d-1} \cap \mathbb{N}^d$$

with two vertices adjacent iff they differ in exactly two coordinates.

Simplicial Rook Graphs

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Simplicial Rook Graphs

- $|V(d,n)| = v = \binom{n+d-1}{d-1}$
- $SR(d,n)$ is regular of degree $\delta = (d-1)n$

- Eigenspaces of adjacency matrix A and Laplacian matrix L are the same because $AX = \lambda X \iff LX = (\delta - \lambda)X$

- Independence number $\alpha(SR(d,n))$ = maximum number of nonattacking rooks on a simplicial chessboard

- $\alpha(SR(3,n)) = \lfloor (2n+3)/3 \rfloor$
  [Nivasch–Lev 2005; Blackburn–Paterson–Stinson 2011]

The Adjacency and Laplacian Matrices

Adjacency matrix of a graph G: $A = A(G)$ = matrix with rows and columns indexed by $V(G)$ with 1s for edges, 0s for non-edges

Laplacian matrix of G: $L = D - A$, where D = diagonal matrix of vertex degrees

- A acts on the vector space $\mathbb{R}V$ by

$$A\mathbf{v} = \sum_{\text{neighbors } \mathbf{w} \text{ of } \mathbf{v}} \mathbf{w}$$

- Eigenvalues of $A, L \implies$ connectivity, spanning trees, ...
- G regular $\implies$ eigenspaces of A, L are the same

The Spectrum of $A(3, n)$

Theorem (JLM/JDW, 2012)

The eigenvalues of $A(3,n) = A(SR(3,n))$ are as follows:

$n = 2m+1$ odd		$n = 2m$ even	
Eigenvalue	**Multiplicity**	**Eigenvalue**	**Multiplicity**
-3	$\binom{2m}{2}$	-3	$\binom{2m-1}{2}$
$-2, \ldots, m-3$	3	$-2, \ldots, m-4$	3
$m-1$	2	$m-3$	2
$m, \ldots, n-2$	3	$m-1, \ldots, n-2$	3
$2n$	1	$2n$	1

Method of proof: Construct explicit eigenvectors.

Corollary

The number of spanning trees of $SR(3, n)$ *is*

$$\begin{cases} \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)^2(n+2)(3n+5)^3} & \textit{if } n \textit{ is odd,} \\[2em] \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)(n+2)^2(3n+4)^3} & \textit{if } n \textit{ is even.} \end{cases}$$

Simplicial Rook Graphs in Arbitrary Dimension

Conjecture

The graph $SR(d, n)$ is integral for all d and n.

Partial results for least eigenvalue λ and corresp. eigenspace W:

- Eigenvectors come from lattice permutohedra.
- If $n \geq \binom{d}{2}$, then $\lambda = -\binom{d}{2}$ and $\dim W = \binom{n-(d-1)(d-2)/2}{d-1}$. Note that
$$\lim_{n\to\infty} \frac{\dim W}{|V(d,n)|} = 1.$$
- If $n < \binom{d}{2}$, then the least eigenvalue appears to be $-n$, and $\dim W$ is the Mahonian number $M(d, n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions.

Hexagon Vectors in $V(3, n)$

For each "internal" vertex $\mathbf{v} \in V(3, n)$ (i.e., $v_i > 0$ for all i), the signed characteristic vector of the hexagon centered at $\mathbf{v}$ is an eigenvector with eigenvalue -3.

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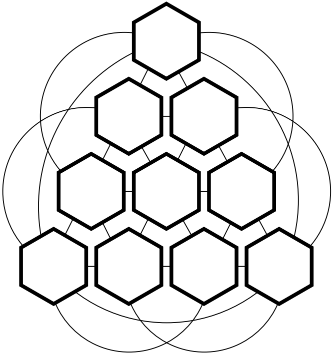

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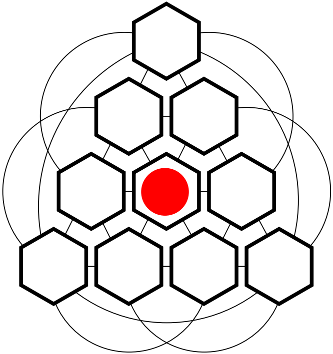

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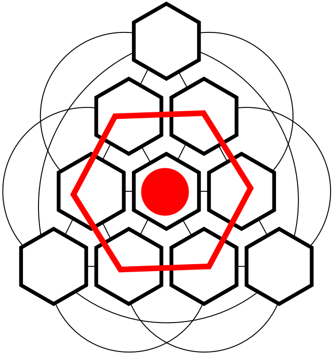

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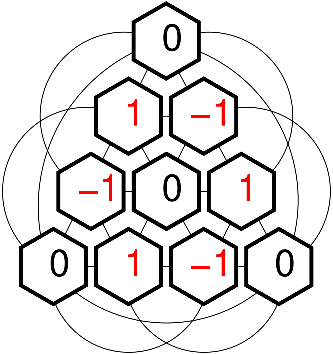

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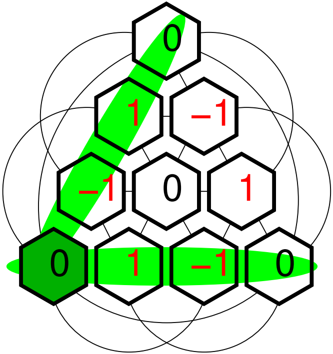

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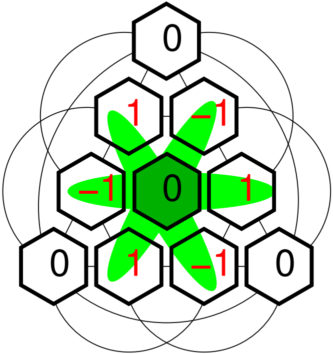

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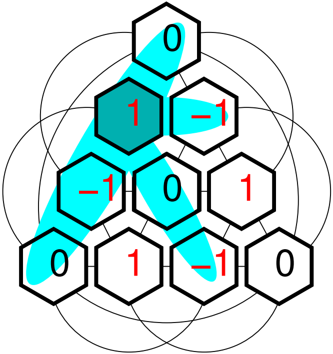


Hexagon Vectors in $V(3, n)$

- Number of possible centers for a hexagon vector = number of interior vertices of $n\Delta^{d-1}$ =

$$\binom{v-1}{2}.$$

- The hexagon vectors are all linearly independent.

- The other $\binom{v+2}{2} - \binom{v-2}{2} = 3v$ eigenvectors have explicit formulas in terms of characteristic vectors of lattice lines.

Permutohedron Vectors in $G(d, n)$

Definition

Let $\mathbf{p} \in \mathbb{Z}^d$ (if d is odd) or $(\mathbb{Z} + \frac{1}{2})^d$ (if d is even). The lattice permutohedron centered at $\mathbf{p}$ is

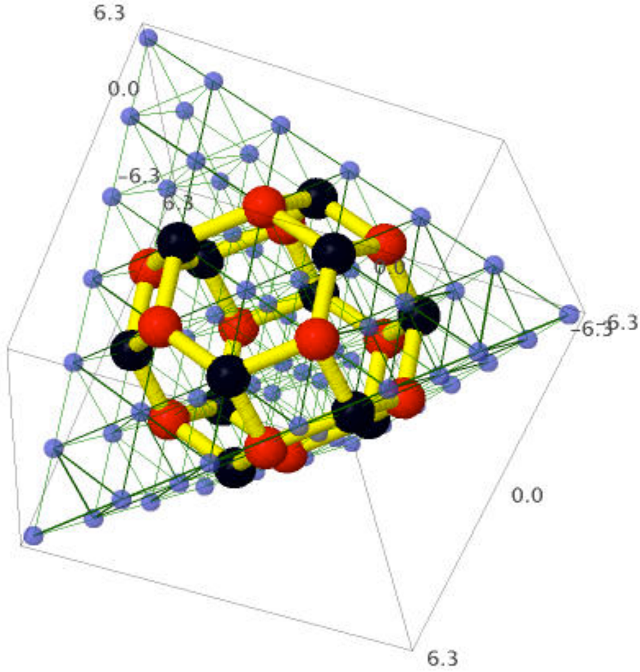
$$\mathrm{Per}(\mathbf{p}) \;=\; \{\mathbf{p} + \sigma(\mathbf{w}) \colon \sigma \in \mathfrak{S}_d\}$$

where $\mathfrak{S}_d$ is the symmetric group and

$$\mathbf{w} \;=\; \left(\frac{1-d}{2},\ \frac{3-d}{2},\ \ldots,\ \frac{d-3}{2},\ \frac{d-1}{2}\right).$$

"Most" eigenvectors of $SR(d, n)$ are signed characteristic vectors $\mathcal{H}_\mathbf{p}$ of lattice permutohedra inscribed in the simplex $n\Delta^{d-1}$.

[SHOW THE NIFTY SAGE PICTURE]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Permutohedron Eigenvectors

- Each $\mathcal{H}_{\mathbf{p}}$ is an eigenvalue of $A(d,n)$ with eigenvalue $-\binom{d}{2}$
- The $\mathcal{H}_{\mathbf{p}}$ are linearly independent.
- Permutohedron vectors account for "most" eigenvectors:

$$\frac{\#\{\mathbf{p}\colon \operatorname{Per}(\mathbf{p}) \subset V(d,n)\}}{|V(d,n)|} = \frac{\binom{n-\binom{d-1}{2}}{d-1}}{\binom{n+d-1}{d-1}} \;\to\; 1 \quad \text{as } n \to \infty.$$

The Case $n < \binom{d}{2}$

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Number of partial permutohedra $=$ Mahonian number $M(d, n)$
$=$ number of permutations in $\mathfrak{S}_d$ with n inversions
$=$ coefficient of q^n in $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{d-1})$

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Construction uses (ordinary, non-simplicial) rook theory!

The Case $n < \binom{d}{2}$

- Permutation $\pi \in \mathfrak{S}_d$ with n inversions $\rightarrow$ "inversion word" $(a_1, \ldots, a_d)$, where $a_i = \#\{j \in [d] : \ \pi_i > \pi_j\}$ (note that $\sum a_i = n$)
- Rook placement σ on skyline Ferrers board $(a_1, \ldots, a_d) \rightarrow$ lattice point $x(\sigma) = (a_i + i - \sigma_i) \in n\Delta^{d-1}$
- Eigenvector $X_\pi = \sum_\sigma \varepsilon(\sigma) x(\sigma)$
- Proof that X_π is an eigenvector: sign-reversing involution moving rooks around

Open Problems

- (The big one.) Prove that $A(d, n)$ (equivalently, $L(d, n)$) has integral spectrum for all d, n. (Verified for lots of d, n.)
- The induced subgraphs

$$SR(d, n)|_{V(d,n) \cap \mathrm{Per}(\mathbf{p})}$$

  also appear to be Laplacian integral for all $d, n, \mathbf{p}$. (Verified for $d \leq 6$.)
- Is $A(d, n)$ determined up to isomorphism by its spectrum? (We don't know.)

Acknowledgements

Thanks to...

- The MathOverflow crowd (in particular Cristi Stoica and Noam Elkies)
- Sage (sagemath.org)
- Mahir, Michael and Jeff for organizing
- You for listening!

Preprint: arxiv:1209.3493