Simplicial and Cellular Spanning Trees

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Graphs

A **graph** is a pair $G = (V, E)$, where

- V is a set of **vertices**, and
- E is a set of **edges**, each joining two vertices (its **endpoints**).

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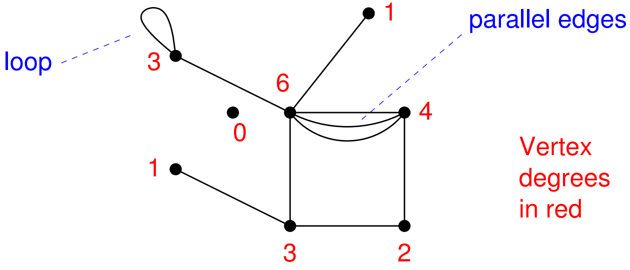
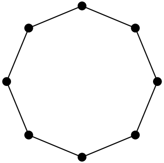
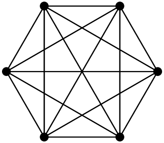
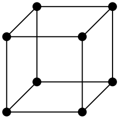




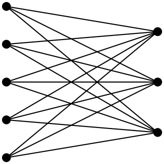
Cycle graph C_8



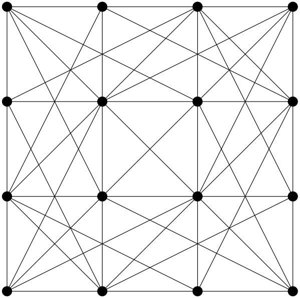
Complete graph K_6



Cube graph Q_3



Complete bipartite graph $K_{5,3}$



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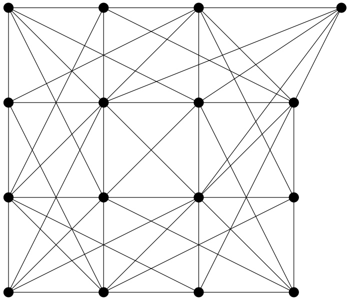

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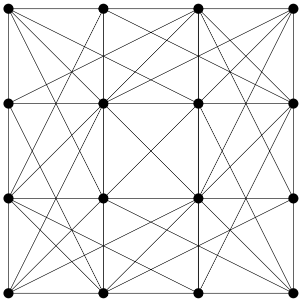
Spanning Trees

Definition A **spanning tree** of a graph G is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Spanning Trees



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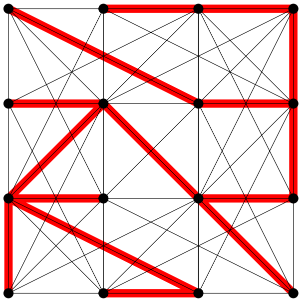

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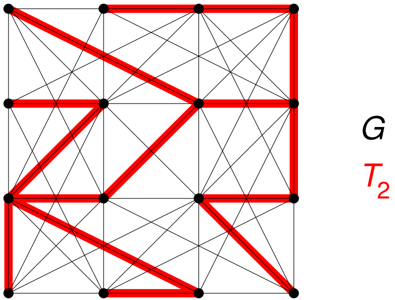
T_1

Spanning Trees



Counting Spanning Trees

$\mathscr{T}(G)$ = set of spanning trees of G
$\tau(G)$ = number of spanning trees of G

- $\tau(\text{tree}) = 1$
- $\tau(C_n) = n$
- $\tau(K_n) = n^{n-2}$ (Cayley's formula; highly nontrivial!)
- $\tau(K_{m,n}) = n^{m-1} m^{n-1}$
- Many other enumeration formulas for nice graphs

Deletion and Contraction

Let $e \in E(G)$.

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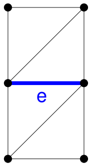

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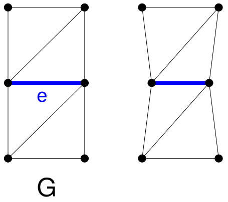

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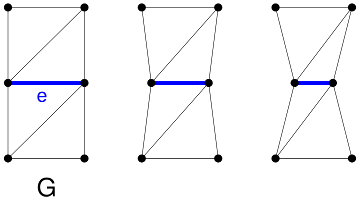

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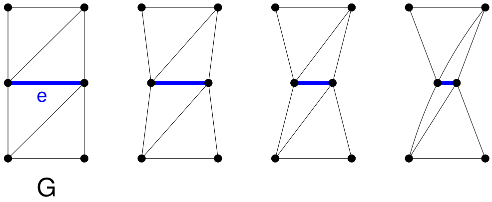

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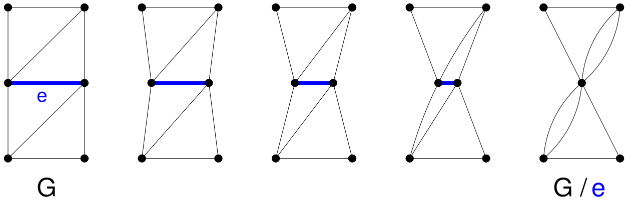

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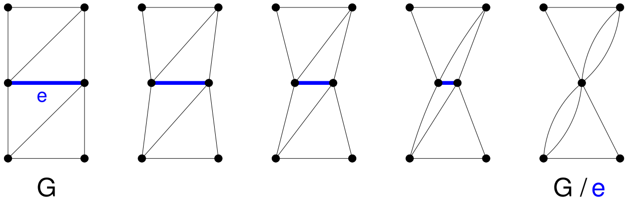

Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$.

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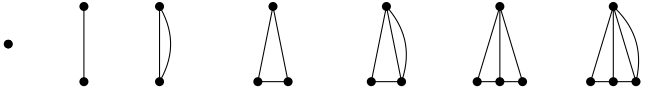
This formula allows easy calculation of $\tau(G)$ and some fun results:

G



$\tau(G)$	1	1	2	3	5	8	13

Unfortunately:

- "easy" does not mean "efficient": $2^{|E|}$ steps are required to calculate $\tau(G)$ this way.
- Useful only for graph families with recursive deletion/contraction structure (*not* K_n, $K_{m,n}$, Q_n, etc.).

The Matrix-Tree Theorem

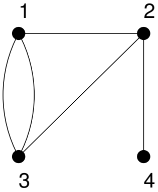
Definition Let G be a connected graph with vertices $[n] = \{1, \ldots, n\}$ and no loops. The **Laplacian** of G is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -(\text{number of edges from } i \text{ to } j) & \text{if } i \neq j. \end{cases}$$

- L is symmetric and positive semi-definite
  - $L = \partial\partial^T$, where $\partial =$ signed vertex-edge incidence matrix
- $\operatorname{rank} L = n - 1$
- $\ker L$ is spanned by the all-1's vector

The Matrix-Tree Theorem

Example

$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} \ .$$

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(2) Let $1 \leq i \leq n$. Form the **reduced Laplacian** L_i by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i \ .$$

The Matrix-Tree Theorem: Proof Sketches

Proof Sketch #1: Use linear algebra and deletion/contraction.

Proof Sketch #2: (Dall–Pfeifle 2014) Dissect one polyhedron with volume $\det L_i$ and reassemble it into one with volume $\tau(G)$. (Ask Ken for details.)

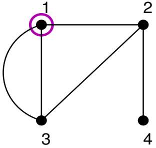
Proof Sketch #3: Let ∂ be the signed vertex/edge incidence matrix of G (so $\operatorname{rank} \partial = n - 1$).

- Note that $L = \partial \partial^T$ and $L_i = \partial_i \partial_i^T$.
- Column bases of ∂ = spanning trees of G.
- Binet-Cauchy:

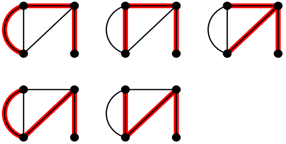
$$\det(\partial_i \partial_i^T) = \sum_{\substack{A \subseteq E(T) \\ |A| = n-1}} (\det \partial_A)^2 = \sum_{T \in \mathscr{T}(G)} (\pm 1)^2 = \tau(G).$$

The Matrix-Tree Theorem: Example

G



$\tau(G) = 5$

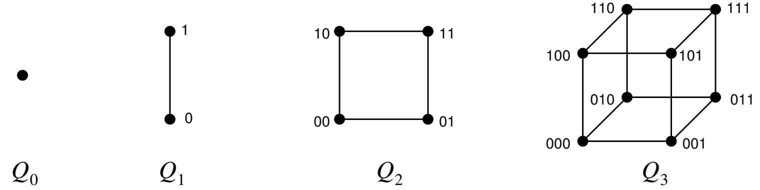


$$\partial = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues: 0, 1, 4, 5

Hypercubes

The **hypercube graph** Q_n has 2^n vertices, labeled by strings of n bits (0's and 1's), with two vertices adjacent if they agree in all but one bit.



Theorem The eigenvalues of the Laplacian of Q_n are $0, 2, 4, \ldots, 2n$, with $2k$ having multiplicity $\binom{n}{k}$. Therefore,

$$\tau(Q_n) = 2^{2^n-n-1} \prod_{k=2}^{n} k^{\binom{n}{k}}.$$

Threshold Graphs

A graph G with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \leq a$ and $b' \leq b$.

Equivalently, the edges of G form an order ideal under componentwise order.

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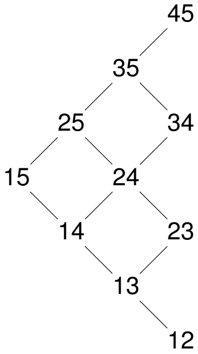

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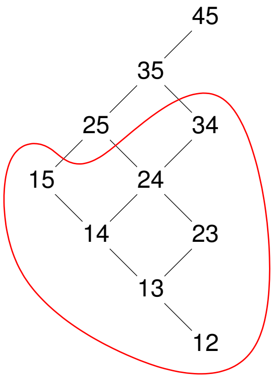

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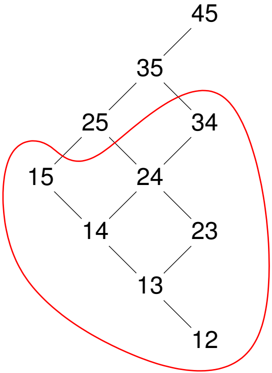

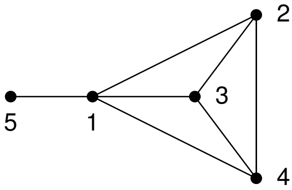

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Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $[n]$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

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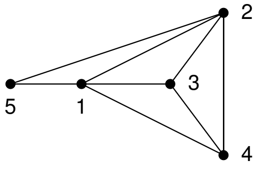

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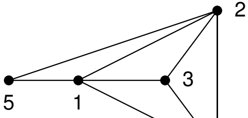

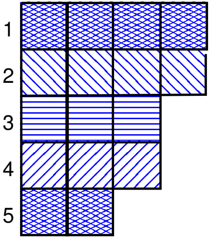

Vertex degrees: 4, 4, 3, 3, 2

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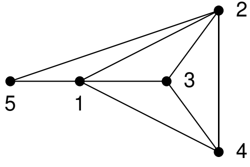

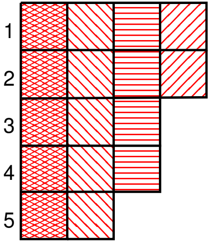

Laplacian eigenvalues: 5, 5, 4, 2, 0

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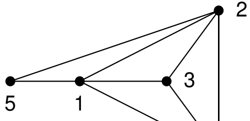
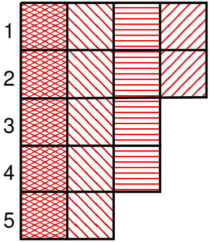




$\tau = 5 \times 4 \times 2 = 40$

Laplacian eigenvalues: 5, 5, 4, 2, 0

Weighted Counting

Theorem [Cayley–Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_1^{\deg_T(1)} \cdots x_n^{\deg_T(n)} \;=\; x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}$$

- Setting $x_i = 1$ for all i recovers $\tau(K_n) = n^{n-2}$
- Can be proved either bijectively (*Prüfer code*) or by a souped-up version of the Matrix-Tree Theorem
- Other weighted tree counting formulas:
  - *Via bijections*: Fiedler-Sedláček (complete bipartite graphs), Knuth, Kelmans, Remmel-Williamson, etc.
  - *Via MTT*: JLM–Reiner (threshold graphs, hypercubes)

Weighted Tree Counts for Threshold Graphs

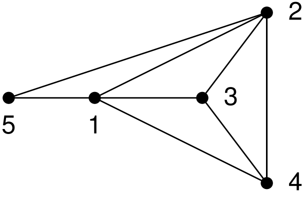
Theorem [JLM–Reiner 2005] Let G be a threshold graph on vertices $[n]$ with degree sequence λ. Weight each edge $e = ij$ with $i < j$ by $x_i y_j$. Then the bidegree generating function is

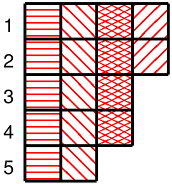
$$\sum_{T \in \mathscr{T}(G)} \prod_{e: i<j} x_i y_j = x_1 y_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_{\min(i,r)} y_{\max(i,r)} \right)$$

and therefore (setting $y_i = x_i$) the degree generating function is

$$\sum_{T \in \mathscr{T}(G)} \prod_{i=1}^{n} x_i^{\deg(i)} = x_1 \cdots x_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_i \right)$$

Weighted Tree Counts for Threshold Graphs





Bidegree generating function:

$$x_1\, y_5\, (x_1 y_2 + x_2 y_2 + x_2 y_3 + x_2 y_4 + x_2 y_5) \times (x_1 y_3 + x_2 y_3 + x_3 y_3 + x_3 y_4)(x_1 y_4 + x_2 y_4)$$

Degree generating function:

$$x_1\, x_2\, x_3\, x_4\, x_5\, (x_1 + x_2 + x_3 + x_4 + x_5)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2)$$

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A ***d*-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.

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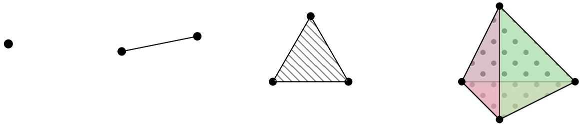

$d=0$ $d=1$ $d=2$ $d=3$

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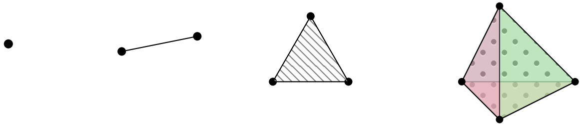

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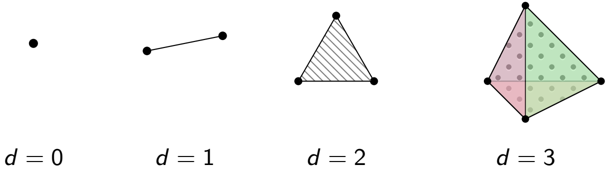

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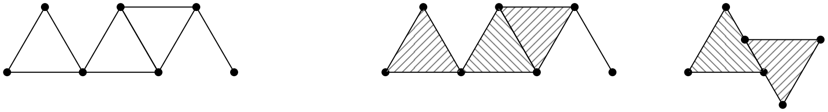

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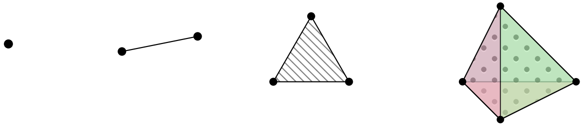

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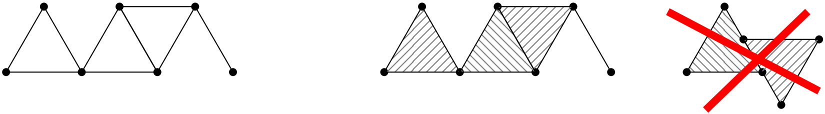
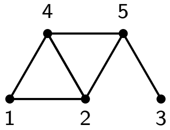
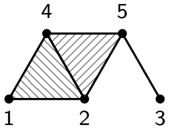


Simplicial Complexes

Combinatorially, a simplicial complex is a set family $\Delta \subseteq 2^{\{1,2,\ldots,n\}}$ such that if $\sigma \in \Delta$ and $\sigma' \subseteq \sigma$, then $\sigma' \in \Delta$.



$\Delta_1 = \langle 12, 14, 24, 24, 25, 35 \rangle$



$\Delta_2 = \langle 124, 245, 35 \rangle$

- **faces** or **simplices**: elements of Δ
- **dimension:** $\dim \sigma = |\sigma| - 1$
- **facet**: a maximal face
- **pure** complex: all facets have equal dimension

Simplicial Spanning Trees

Definition Let Δ^D be a simplicial complex of dimension d.

A subcomplex $\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** (SST) if:

1. Υ contains all simplices of Δ of dimension $< d$.
2. Υ is "acyclic" and "connected".
   - *Technically:* $\tilde{H}_d(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$.
   - *Intuitively:* Υ has no "bubbles" whose boundary is an orientable d- or $(d-1)$-manifold.

As before, we'll write $\mathscr{T}(\Delta)$ for the set of SSTs of Δ.

Examples of SSTs

- $\dim \Delta = 1$: $\mathscr{T}(\Delta) =$ graph-theoretic spanning trees

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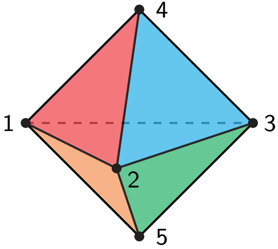
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- If Δ is a simplicial sphere: SSTs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any facet (maximal face)
  - Simplicial spheres are analogous to cycle graphs

Examples of SSTs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



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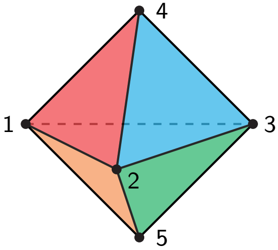

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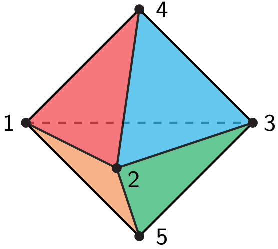

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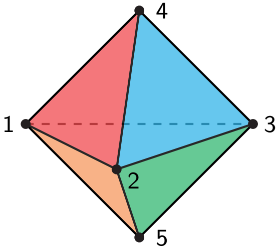

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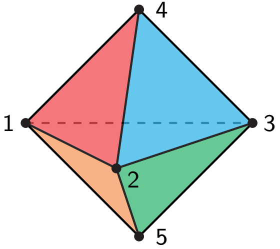

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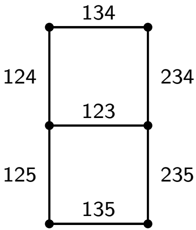

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Simplicial Boundary Maps and Homology

Let Δ be a simplicial complex on vertices $[n]$.
Write Δ_k for the set of k-dimensional faces.

The ***k^{th}* simplicial boundary matrix** of Δ is is

$$\partial_k = \partial_k(\Delta) = [d_{\rho,\sigma}]_{\rho\in\Delta_{k-1},\,\sigma\in\Delta_k}$$

where

$$d_{\rho,\sigma} = \begin{cases} (-1)^j & \text{if } \sigma = \{v_0 < v_1 < \cdots < v_k\} \text{ and } \rho = \sigma \setminus v_j \\ 0 & \text{if } \rho \not\subseteq \sigma \end{cases}$$

Note: ∂_1 is the signed incidence matrix of the 1-skeleton of Δ.

Fact: $\ker\partial_k \supseteq \operatorname{im}\partial_{k+1}$ for all k. (Check it!)

Simplicial Boundary Maps and Homology

Fact: $\ker \partial_k \supseteq \operatorname{im} \partial_{k+1}$ for all k. In other words, the sequence

$$\cdots \to R\Delta_{k+1} \xrightarrow{\partial_{k+1}} R\Delta_k \xrightarrow{\partial_k} R\Delta_{k-1} \to \cdots$$

is a chain complex for any ring R. (Default in this talk: $R = \mathbb{Z}$.)

The homology groups of Δ are

$$\tilde{H}_k(\Delta; R) \;=\; \ker \partial_k \;/\; \operatorname{im} \partial_{k+1}.$$

These are topological invariants of Δ.

- $\tilde{H}_0(\Delta) = 0 \iff \Delta$ is connected
- $\tilde{H}_1(\Delta) = 0 \iff \Delta$ is simply connected (essentially)
- If Δ is contractible then $\tilde{H}_k(\Delta) = 0$ for all k

Simplicial Laplacians

The k^{th} **(updown) Laplacian matrix** of a simplicial complex Δ is

$$L_{k-1}^{\mathrm{ud}}(\Delta) = \partial_k \partial_k^T.$$

- $L_0^{\mathrm{ud}}(\Delta)$ is the usual graph Laplacian.
- $L_{k-1}^{\mathrm{ud}}(\Delta)$ is a square matrix with entries

$$\ell_{\rho,\pi} = \begin{cases} \#\{\sigma \in \Delta_k \ \mid \ \sigma \supseteq \rho\} & \text{if } \rho = \pi, \\ \pm 1 & \text{if } \rho, \pi \text{ lie in a common } k\text{-face}, \\ 0 & \text{otherwise} \end{cases}$$

for $\rho, \pi \in \Delta_{k-1}$.

The Simplicial Matrix-Tree Theorem (Roughly)

Simplicial Matrix-Tree Theorem
(Bolker, Kalai, Adin, Duval–Klivans–JLM, . . .)

Let Δ^d be a simplicial complex.

Form a **reduced Laplacian** $L_T(\Delta)$ from $L(\Delta)$ by deleting the rows and columns corresponding to a $(d-1)$-dimensional SST $T \subseteq \Delta$.

Then the "number" of spanning trees of Δ is $\det L_T$, divided by a correction factor given by T.

The Simplicial Matrix-Tree Theorem (Precisely)

The torsion of a spanning tree $\Upsilon \in \mathscr{T}(\Delta)$ is

$$\left|\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})\right| = \left|\ker \partial_{d-1}(\Upsilon) \ / \ \operatorname{im} \partial_d(\Upsilon)\right|$$

(which must be finite).

- This number is 1 if $\dim \Delta \leq 1$.
- Torsion $\approx$ non-orientability: e.g., $\tilde{H}_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

Simplicial Matrix-Tree Theorem

$$\tau(\Delta) \ \stackrel{\text{def}}{=} \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})|^2 \ = \ \frac{|\tilde{H}_{d-2}(\Delta)|}{|\tilde{H}_{d-2}(T)|} \ \det \hat{L}_T$$

If $d = 1$ then all summands are 1.
In many natural cases, the correction factor is trivial.

Kalai's Theorem

Simplicial generalization of the complete graph:

$$K_{n,d} = \{F \subseteq \{1,\dots,n\} \mid \dim F \leq d\}$$

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Theorem [Kalai 1983]

$$\tau(K_{n,d}) \;=\; n^{\binom{n-2}{d}}.$$

More generally,

$$\sum_{\Upsilon \in \mathscr{T}(K)} |\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{i=1}^{n} x_i^{\deg_\Upsilon(i)} = (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}.$$

Kalai's Theorem

- Kalai's theorem reduces to $\tau(K_n) = n^{n-2}$ when $d = 1$, and the weighted version reduces to Cayley-Prüfer.
- Bolker (1976): Observed that $n^{\binom{n-2}{d}}$ is an exact count of trees for small n, d, but fails for $n = 6$, $d = 2$.
  - The problem is torsion — $\mathbb{RP}^2$ requires six vertices to triangulate
- Adin (1992): Analogous formula for **complete colorful complexes**, generalizing $\tau(K_{n,m}) = n^{m-1} m^{n-1}$

Shifted Simplicial Complexes

A simplicial complex Δ with vertex set $\{1, 2, \ldots, n\}$ is **shifted** if whenever $a_1 a_2 \cdots a_k \in \Delta$ and $b_i \leq a_i$ for all i, then $b_1 b_2 \cdots b_k \in \Delta$.

(So one-dimensional shifted complexes are just threshold graphs.)

Theorem [Duval–Reiner 2002]

Let λ_i = number of max-dim faces containing i.
Then eigenvalues of $L(\Delta)$ = column lengths of λ.
(Generalization of Merris' Theorem)

Theorem [Duval–Klivans–JLM 2009]

Factorization of multidegree g.f. for spanning trees of a shifted complex. (Generalization of JLM–Reiner formula)

Further Directions

- Theory of SSTs and the Matrix-Tree Theorem generalize easily from simplicial complexes to cell complexes
  - Cubes and their skeletons [Duval–Klivans–JLM 2011], [Aalipour–Duval–Kook–Lee–JLM 2017^{+}]
  - Cellular MTT discovered independently in contexts of probability [Lyons 2009] and mathematical physics [Catanzaro–Chernyak–Klein 2015]
- Simplicial/cell complexes that have integer Laplacian eigenvalues "should" have factorizable weighted tree g.f.'s
  - Matroid complexes; others?
- Critical groups:
  - Complex $\Delta \Rightarrow$ abelian group $K(\Delta)$ of size $\tau(\Delta)$
  - Cuts, flows, sandpile theory, "algebraic geometry on graphs"
  - Group structure very mysterious