Unbounded Matroids

Jeremy L. Martin
Department of Mathematics
University of Kansas

Triangle Lectures in Combinatorics
UNC Greensboro
December 3, 2022

Coauthors

Jonah Berggren
(Kentucky)

José Samper
(PUC–Chile)

Matroids

Matroids are combinatorial models of linear (in)dependence.
A matroid M on finite ground set E can be characterized by its **bases**, or its **rank function**, or its **lattice of flats**, or ...

Definition

A **matroid basis system** is a nonempty set family $\mathcal{B} \subset 2^E$ with

1. $|B| = |B'| = r$ for all $B, B' \in \mathcal{B}$
2. $\forall e \in B \backslash B' \colon\ \exists e' \in B' \backslash B \colon\ B \backslash e \cup e' \in \mathcal{B}$ (**exchange axiom**)

Canonical example #1: $E =$ vectors, $\mathcal{B} =$ bases of their span
Canonical example #2: $E = E(G)$, $\mathcal{B} =$ spanning trees

- Intuition: There are lots of ways of getting from B to B' by changing one element at a time.
- B, B' are "close" only if $|B \triangle B'| = 2$.

Matroids

Rank function: $\rho : 2^E \to \mathbb{N}$ satisfying

- $\rho(A) \leq |A|$;
- $A \subseteq B \implies \rho(A) \leq \rho(B)$ (**monotonicity**);
- $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$ (**submodularity**).

"Cryptomorphisms" between basis system and rank function:

$$\mathcal{B} = \{B \subseteq E : \rho(B) = |B| = \rho(E)\}$$
$$\rho(A) = \max\{|A \cap B| : B \in \mathcal{B}\}$$

Flats: $S \subseteq E$ such that $T \supsetneq S \implies \rho(T) > \rho(S)$

- Flats form a geometric lattice $\mathcal{L}(M)$

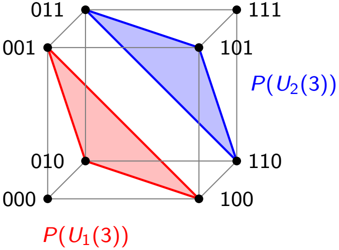
Matroid Base Polytopes

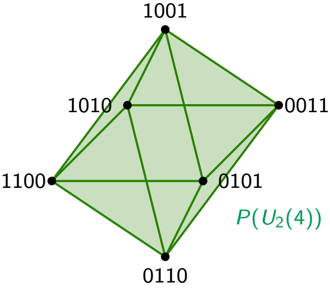
The **base polytope** of a matroid M^E is

$$P(M) = \text{conv}\{\chi_B \mid B \in \mathcal{B}(M)\} \subset [0,1]^E$$

Example

M = **uniform matroid** $U_r(n)$ with $\mathcal{B} = \binom{[n]}{r}$
$P(M) = \{\mathbf{x} \in [0,1]^n \mid \sum x_i = r\}$ (**hypersimplex**)





Matroid Base Polytopes: Basic Properties

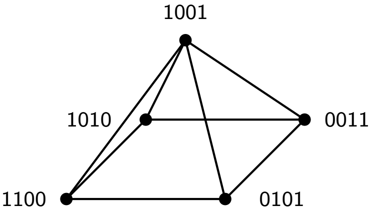
For $\mathbf{x} \in \mathbb{R}^E$ and $A \subseteq E$, write $\mathbf{x}(A) = \sum_{i \in A} x_i$.

- $P(M)$ lies in the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}(E) = \rho(E)\}$
  - In particular, $\dim P(M) < |E|$.
- $P(M \oplus M') = P(M) \times P(M')$
  - Here $\mathcal{B}(M \oplus M') = \{B \cup B' : \; B \in \mathcal{B}(M), \; B' \in \mathcal{B}(M')\}$
- Inequality description [Edmonds '70]:

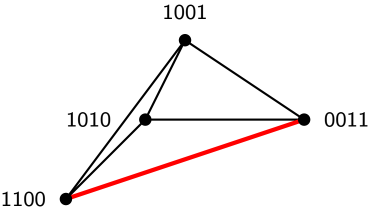
$$P(M) = \{\mathbf{x} \in H \mid \mathbf{x}(A) \leq \rho(A) \;\; \forall A \subseteq E\}$$

  - $A \subseteq \mathcal{L}(M)$ suffices. Facets: [Feichtner–Sturmfels '05]

- Vertices of $P(M) \longleftrightarrow$ bases $B \in \mathcal{B}(M)$
- Edges of $P(M) \longleftrightarrow$ basis exchanges



$\mathcal{B} = \{12, 13, 14, 24, 34\}$



$\mathcal{B} = \{12, 13, 14, 34\}$
Not a matroid base polytope
($\mathcal{B}$ fails exchange condition)

Generalized Permutahedra

Matroid base polytopes are **generalized permutahedra** [Postnikov '09]:

- all edges are parallel to vectors $\mathbf{e}_i - \mathbf{e}_j$ (= type-A roots)
- Normal fan coarsens the braid fan
- Face maximized by a linear functional $\mathbf{x} \to \mathbf{c} \cdot \mathbf{x}$ depends only on the relative order of $c_1, \ldots, c_n$

Matroid base polytopes are **exactly** the GPs with 0,1-vertices [Gel'fand–Goresky–Macpherson–Serganova '87]

Polymatroids

Definition (Edmonds '70)

A **polymatroid rank function** is a submodular rank function $\rho : 2^E \to \mathbb{R}$ that is

- *calibrated*: $\rho(\emptyset) = 0$,
- *monotone:* $S \subseteq T \implies \rho(S) \leq \rho(T)$,
- but does not necessarily satisfy $\rho(S) \leq |S|$.

The **base polytope** of ρ is

$$P(M) = \left\{ \mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}(A) \leq \rho(A) \ \forall A \subseteq E, \ \ \mathbf{x}(E) = \rho(E) \right\}.$$

This construction gives a bijection between generalized permutahedra and polymatroids.

Submodular Systems

Definition (Fujishige '05)

A **submodular system** is a triple $M = (E, \mathcal{D}, \rho)$, where

- $\mathcal{D}$ is a distributive sublattice of 2^E; and
- $\rho : \mathcal{D} \to \mathbb{R}$ is a calibrated submodular rank function.

(Or: $\rho : 2^E \to \mathbb{R} \cup \{\infty\}$ and $\mathcal{D} = \{A \subseteq E \mid \rho(A) < \infty\}$.)

The corresponding **base polyhedron** is (again)

$$P(M) = \left\{ \mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}(A) \leq \rho(A) \ \forall A \in \mathcal{D}, \ \ \mathbf{x}(E) = \rho(E) \right\}.$$

This polyhedron is unbounded iff $\mathcal{D} \neq 2^E$. It is a generalized permutahedron: all edges and rays are parallel to roots of type A.

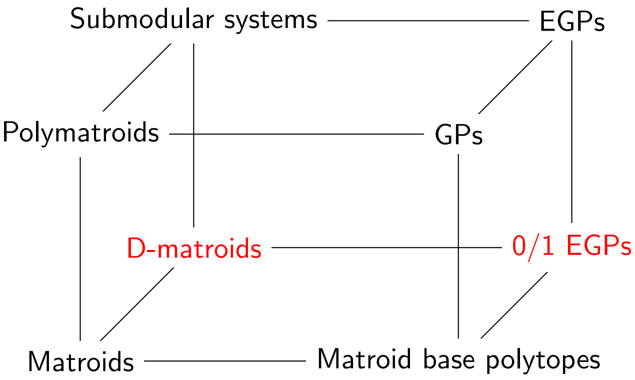
D-Matroids

Our project: Study 0/1-generalized permutahedra that need not be bounded (unbounded matroid polyhedra) and their combinatorial analogues (unbounded matroids/D-matroids).

Definition

A **D-matroid** is a submodular system $M = (E, \mathcal{D}, \rho)$, where $\mathcal{D} \subseteq 2^E$ is a distributive lattice and $\rho : \mathcal{D} \to \mathbb{N}$ is **integral**, **monotone**, and **unit-increase** (as well as calibrated and submodular).

- D-matroids are essentially identical to the *pregeometries* of [Faigle 1980]. However, Faigle defined bases differently (and purely combinatorially).
- A D-matroid is a matroid precisely when $\mathcal{D} = 2^E$.
- D-matroids admit a Hopf monoid structure [Castillo–JLM–Samper '22$^+$]

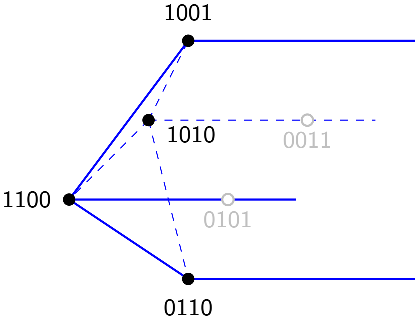


- Horizontal lines are bijections; others are inclusions
- Left/right = combinatorial/geometric
- Bottom/top = integer/real
- Front/back = bounded/possibly unbounded

Example: The Stalactite

The **stalactite** is the polyhedron

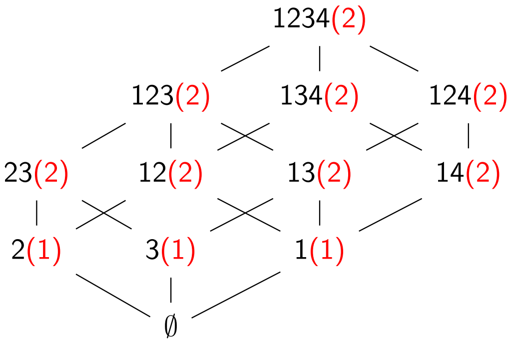
$$Q = \{\mathbf{x} \in \mathbb{R}^4 \colon x_1 + x_2 + x_3 + x_4 = 2,\ x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3 \leq 1\}.$$



- Recession cone: $R = \mathbb{R}_{\geq 0}\langle(-1, 0, 0, 1)\rangle$
- $\mathbf{e}_4 - \mathbf{e}_1 \in R \iff 1 <_{\mathcal{P}} 4$, where $\mathcal{P} = \mathrm{Irr}(\mathcal{D})$

$\mathcal{D}$

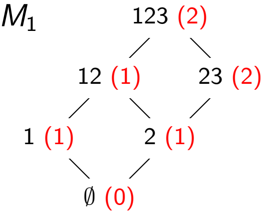
$\rho(A) = \min(|A|, 2)$

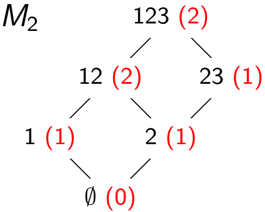


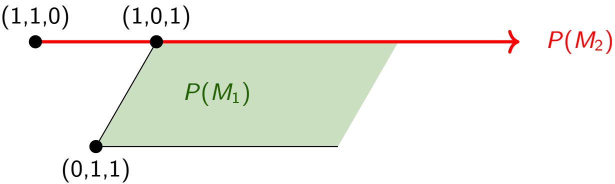
Maximal chain in $\mathcal{D}$	$\rightsquigarrow$	**Vertex** $\mathbf{x} = (x_1, \ldots, x_n)$
$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = E$		$x_{A_i \setminus A_{i-1}} = \rho(A_i) - \rho(A_{i-1})$

- Vertices $\{12, 13, 14, 23\}$ do not form a matroid basis system

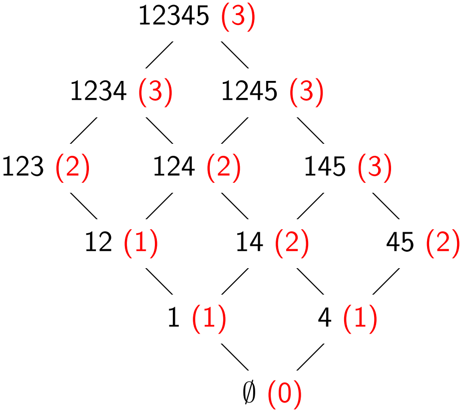
More Examples







More Examples



- Minimal elements of maximum rank are not all same size
- Bases: 134, 145 (note that 134 $\notin \mathcal{D}$)

Lattice Extensions and Sheared Polyhedra

Definition

Let $M = (E, \mathcal{D}, \rho)$ be a D-matroid and $\mathcal{D}'$ be a distributive lattice with $\mathcal{D} \subseteq \mathcal{D}' \subseteq 2^E$.

A **lattice extension** of M is a D-matroid $M' = (E, \mathcal{D}', \rho')$ such that $\rho'|_{\mathcal{D}} = \rho$.

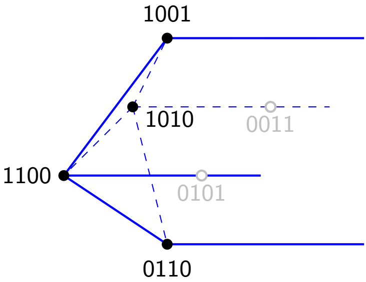
Theorem (Berggren–JLM–Samper)

M' is a lattice extension of M if and only if

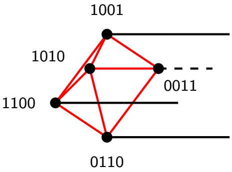
- *$P(M') \subseteq P(M)$ and*
- *$V(P(M')) \supseteq V(P(M))$.*

In this case we say that $P(M')$ is a **shearing** of $P(M)$.

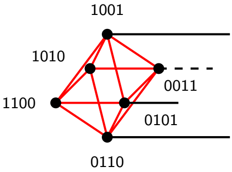
Example: Shearing the Stalactite



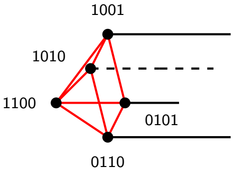
$$\rho(A) = \min(|A|, 2)$$



$\rho(A) = \min(|A|, 2)$
except $\rho(24) = 1$



$\rho(A) = \min(|A|, 2)$



$\rho(A) = \min(|A|, 2)$
except $\rho(34) = 1$

Generous Extensions

Theorem (Berggren–JLM–Samper)

Let $M = (E, \mathcal{D}, \rho)$ be a D-matroid and $\mathcal{D}'$ be a distributive lattice with $\mathcal{D} \subseteq \mathcal{D}' \subseteq 2^E$.

Then there exists a D-matroid $M' = (E, \mathcal{D}', \rho')$ (the **generous extension of** M **to** $\mathcal{D}'$*) such that:*

1. *M' is a lattice extension of M to $\mathcal{D}'$.*
2. *If $(E, \mathcal{D}', \phi)$ is any lattice extension of M to $\mathcal{D}'$, then $\rho'(A) \geq \phi(A)$ for all $A \subseteq E$.*

Corollary

1. *$P(M')$ is the unique largest sheared polyhedron of $P(M)$ with recession cone $R(\mathcal{D}')$.*
2. *$P(M)$ contains a unique largest matroid base polytope.*

Generous Extensions

Sketch of proof: It is enough to consider the case $\mathcal{D}' = \mathcal{D}[a] =$ smallest distrib. lattice containing $\mathcal{D} \cup \{\{a\}\}$, where $a \in E$. Define

$$\rho'(S) = \begin{cases} \rho(S) & \text{if } S \in \mathcal{D} \\ \rho(S-a) & \text{if } S \notin \mathcal{D} \text{ and } \rho(S-a) = \rho(\sup_{\mathcal{D}}(S)) \\ \rho(S-a)+1 & \text{if } S \notin \mathcal{D} \text{ and } \rho(S-a) < \rho(\sup_{\mathcal{D}}(S)) \end{cases}$$

where $\sup_{\mathcal{D}}(S) =$ smallest element of $\mathcal{D}$ containing S.

("Tacking on a increments rank except when it obviously can't.")

Every extension to $\mathcal{D}'$ is bounded by ρ'; as a consequence, the order of adjoining atoms does not matter.

Generous Extensions

Corollary

1. *Every D-matroid base polyhedron is the Minkowski sum of a matroid base polytope with its recession cone.*
2. *The base polytope of the generous matroid extension of a D-matroid M is*
   - *the convex hull of all 0,1-vectors in $P(M)$;*
   - *the intersection of $P(M)$ with the appropriate hypersimplex;*
   - *the union of all sheared matroid polytopes in $P(M)$.*

Remark

- We do not know a closed formula for the rank function of the generous matroid extension.
- The construction fails entirely without the $0/1$-condition!

Bases of D-Matroids

Proposition

Let $M' = (E, \mathcal{D}', \rho)$ be a D-matroid with basis system $\mathcal{B}'$ and $\mathcal{D} \subseteq \mathcal{D}'$ a distributive lattice. Let $\mathcal{P} = \mathrm{Irr}(\mathcal{D})$.

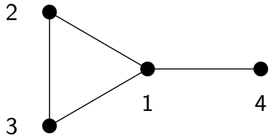
Then the basis system of the lattice restriction $M = M'|_{\mathcal{D}}$ is $\mathrm{im}\, f$, where $f : \mathrm{Lin}(\mathcal{P}) \to \mathcal{B}'$ sends $\sigma \in \mathrm{Lin}(\mathcal{P})$ to the σ-lex-first basis.

Definition

The **pseudo-independence complex** of a D-matroid $M = (E, \mathcal{D}, \rho)$ is the simplicial complex $\Delta(M)$ on E generated by the bases.

Example

The stalactite has $\mathcal{B} = \{12, 13, 14, 23\}$ and $\Delta = \langle 12, 13, 14, 23 \rangle$.



The Pseudo-Independence Complex

Theorem

Every D-matroid pseudo-independence complex $\Delta(M)$ is shellable.

In fact, every generic linear functional ℓ in the interior of $R(P(M))^*$ defines a linear order on vertices of $P(M)$ that is a shelling order on $\Delta(M)$.

Proof uses a polyhedral result of Heaton and Samper.

Questions

- What characterizes these complexes?
- What do their h-numbers count?

D-Matroids and Subspace Arrangements

$\mathcal{A} = (V_1, \ldots, V_n)$ = arrangement of linear subspaces in $\mathbb{k}^d$
$c_i = \text{codim } V_i$

D-matroid $M(\mathcal{A})$ that **represents** $\mathcal{A}$:

$$\begin{aligned} \mathcal{D} &= J([c_1] \times \cdots \times [c_n]) \\ &= \{\mathbf{a} = (a_1, \ldots, a_n) \mid 0 \le a_i \le c_i \quad \forall i\} \end{aligned}$$

$$\rho(\mathbf{a}) = \max \left\{ \text{codim}(W_1 \cap \cdots \cap W_n) \;\middle|\; \begin{array}{c} V_i \subseteq W_i \subseteq \mathbb{k}^d \\ \text{codim } W_i = a_i \end{array} \quad \forall i \right\}$$

- This construction is due to [Barnabei–Nicoletti–Pezzoli '98].
- $M(\mathcal{A})$ is a *poset matroid* in their sense (a D-matroid such that every vertex is in $\mathcal{D}$).

Theorem

1. *Suppose that* $c_k = \operatorname{codim} V_k \geq 2$ *for some* $k \in [n]$*. Let a be the atom corresponding to the top element of the chain* $[0, c_k]$*. Then the generous extension of* $M(\mathcal{A})$ *to* $\mathcal{D}[a]$ *represents*

$$\mathcal{A} \setminus \{V_k\} \cup \{V_k', V_k''\}$$

where V_k', V_k'' *are generic linear spaces containing* V_k *of codimensions 1 and* $c_k - 1$*.*

2. *The generous matroid extension of* $M(\mathcal{A})$ *represents any hyperplane arrangement formed by replacing every* V_i *with* c_i *generic hyperplanes containing* V_i*.*

In particular, generous matroids are **multisymmetric** in the sense of [Crowley–Huh–Larson–Simpson–Wang '22$^+$].

Thank you!

References

M. Barnabei, G. Nicoletti, and L. Pezzoli. Matroids on partially ordered sets. *Adv. in Appl. Math.*, 21(1):78–112, 1998.

F. Castillo, J.L. Martin, and J.A. Samper. Hopf monoids of ordered simplicial complexes. Preprint, arXiv:2011.14955, 2020.

C. Crowley, J. Huh, M. Larson, C. Simpson, and B. Wang. The Bergman fan of a polymatroid. Preprint, arXiv:2207.08764, 2022.

J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial optimization—Eureka, you shrink!*, *Lect. Notes Comput. Sci.*, 2570, pp. 11–26, 2003.

U. Faigle. Geometries on partially ordered sets. *J. Combin. Theory Ser. B*, 28(1):26–51, 1980.

E.M. Feichtner and B. Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)* 62(4):437–468, 2005.

S. Fujishige. *Submodular functions and optimization*, 2nd edn., Elsevier, 2005.

I.M. Gel'fand, R.M. Goresky, R.D. MacPherson, and V.V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. Math.*, 63(3):301–316, 1987.

A. Heaton and J.A. Samper. Dual matroid polytopes and internal activity of independence complexes. Preprint, arXiv:2005.04252, 2020.

A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.