Oscillation estimates of eigenfunctions via the combinatorics of noncrossing partitions

Vera M. Hur (University of Illinois)
Mathew A. Johnson (University of Kansas)
Jeremy L. Martin (University of Kansas)

SIAM Central States Section Meeting
University of Oklahoma
October 6, 2018

Combinatorics and PDE:
A Love Story

Vera M. Hur (University of Illinois)
Mathew A. Johnson (University of Kansas)
Jeremy L. Martin (University of Kansas)

SIAM Central States Section Meeting
University of Oklahoma
October 6, 2018

The Problem

PDE Setup: Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is an eigenfunction for the *fractional Schrödinger operator*

$$H_\alpha = \underbrace{\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}}_{\text{fractional Laplacian}} + V(x)$$

where $0 < \alpha < 2$ and $V : \mathbb{R} \to \mathbb{R}$ is some potential function.

(Note that $\alpha = 2$ gives the classical Laplacian).

Problem: Determine, or at least bound, the number of times ϕ changes sign (relevant in stability theory.)

Motivation

The fractional Laplacian appears in several PDEs related to wave motion:

$$u_t + u_x + \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u_x + f(u)_x = 0 \qquad \text{(fKdV)}$$

$$u_t + u_x + \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u_t + f(u)_x = 0 \qquad \text{(fBBM)}$$

$$iu_t - \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u + f(|u|)u = 0 \qquad \text{(fNLS)}$$

Classical ($\alpha = 2$): arise in water wave theory; NLS also in nonlinear optics, Bose-Einstein condensates, etc.
Fractional ($0 < \alpha < 2$): changes dispersion

From The Line To The Half-Plane

Suppose that the $L^2(\mathbb{R})$ spectrum of the fractional Schrödinger operator H_α has at least N eigenvalues (with multiplicity):

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_N$$

with corresponding eigenfunctions $\phi_1, \ldots, \phi_N \in H^{\alpha/2}(\mathbb{R}) \cap C^0(\mathbb{R})$.

Each ϕ gives rise to a classical boundary value problem on the upper half-plane $\mathbb{P}^2$:

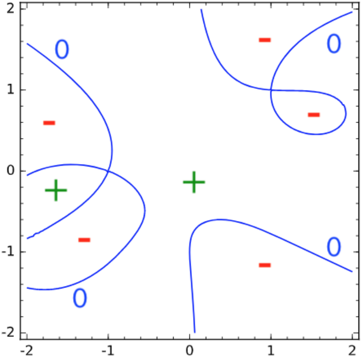
$$\begin{cases} \nabla \cdot (y^{1-\alpha} \nabla w) = 0 & \text{for} \quad (x, y) \in \mathbb{P}^2, \\ w = \phi & \text{on} \quad \partial\mathbb{P}^2 = \mathbb{R} \times \{0\}, \end{cases}$$

which admits a unique solution $E(\phi)$.

Nodal Domains

Courant's Nodal Domain Theorem: For each $n = 1, 2, \ldots, N$, the extension $E(\phi_n)$ has at most n **nodal domains**.

Nodal domains of a continuous function $f : X \to \mathbb{R}$
$= $ connected components of $\{x \in X : \ f(x) \neq 0\}$
$= $ maximal subsets of X on which f has constant sign



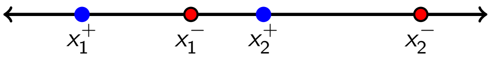
Nodal Domains and Sign Changes

Theorem [Frank and Lenzmann 2013]
The second eigenfunction ϕ_2 of H_α changes sign at most twice.

Nodal Domains and Sign Changes

Theorem [Frank and Lenzmann 2013]
The second eigenfunction ϕ_2 of H_α changes sign at most twice.

Proof. Suppose $x_1^+ < x_1^- < x_2^+ < x_2^- \in \mathbb{R}$ such that ϕ_2 is positive at each x_i^+ and negative at each x_i^-.

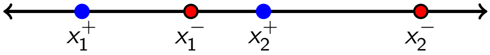


Nodal Domains and Sign Changes

Theorem [Frank and Lenzmann 2013]
The second eigenfunction ϕ_2 of H_α changes sign at most twice.

Proof. Suppose $x_1^+ < x_1^- < x_2^+ < x_2^- \in \mathbb{R}$ such that ϕ_2 is positive at each x_i^+ and negative at each x_i^-.

By Courant's theorem, $E(\phi_2)$ has at most two, hence exactly two nodal domains: $U^+ \supset \{x_1^+, x_2^+\}$ and $U^- \supset \{x_1^-, x_2^-\}$.

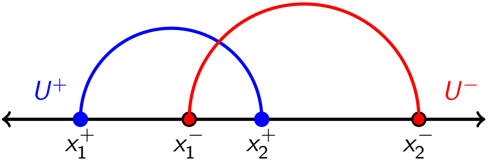


Nodal Domains and Sign Changes

Theorem [Frank and Lenzmann 2013]
The second eigenfunction ϕ_2 of H_α changes sign at most twice.

Proof. Suppose $x_1^+ < x_1^- < x_2^+ < x_2^- \in \mathbb{R}$ such that ϕ_2 is positive at each x_i^+ and negative at each x_i^-.

By Courant's theorem, $E(\phi_2)$ has at most two, hence exactly two nodal domains: $U^+ \supset \{x_1^+, x_2^+\}$ and $U^- \supset \{x_1^-, x_2^-\}$.

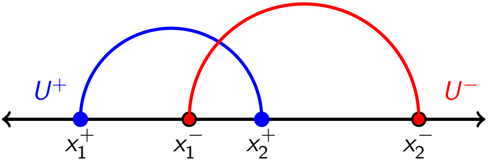


Nodal Domains and Sign Changes

Theorem [Frank and Lenzmann 2013]
The second eigenfunction ϕ_2 of H_α changes sign at most twice.

Proof. Suppose $x_1^+ < x_1^- < x_2^+ < x_2^- \in \mathbb{R}$ such that ϕ_2 is positive at each x_i^+ and negative at each x_i^-.

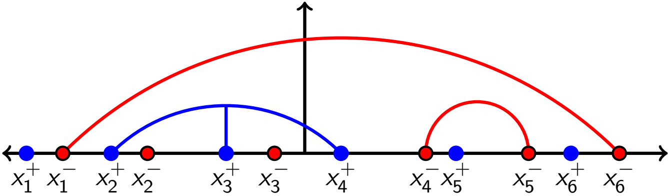
By Courant's theorem, $E(\phi_2)$ has at most two, hence exactly two nodal domains: $U^+ \supset \{x_1^+, x_2^+\}$ and $U^- \supset \{x_1^-, x_2^-\}$.



This can't happen. ("*Don't cross the streams!*")

Higher Eigenfunctions

Goal: Generalize the Frank–Lenzmann approach to higher eigenfunctions.
If ϕ has $2k - 1$ sign changes, the picture might look like this:



Which roots lie in common nodal domains can be modeled combinatorially as a **noncrossing partition**.

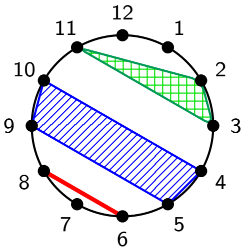
Noncrossing Partitions

Definition

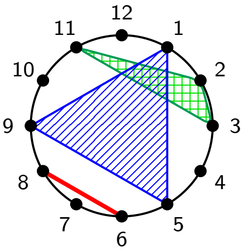
A **partition** of a totally ordered set X is the family of equivalence classes ("blocks") of an equivalence relation $\sim$ on X.

A partition is **noncrossing** if

$$i < j < k < \ell, \quad i \sim k, \quad j \sim \ell \quad \implies \quad i \sim j \sim k \sim \ell.$$



Noncrossing partition

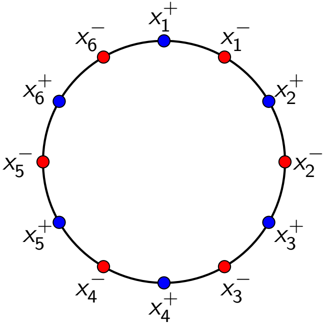


1,3,5,11 is a crossing

Monochromatic Noncrossing Partitions

Definition

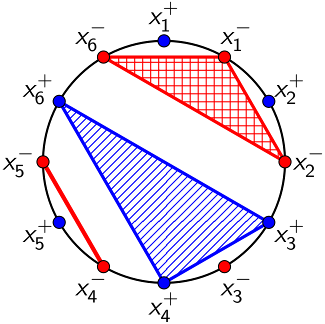
A noncrossing partition of $[2n]$ is **monochromatic** if no block has both even and odd elements.



Monochromatic Noncrossing Partitions

Definition

A noncrossing partition of $[2n]$ is **monochromatic** if no block has both even and odd elements.



An MNP models (some of) the nodal domain structure for an extension $E(\phi)$ of an (eigen)function $\phi : \mathbb{R} \to \mathbb{R}$ with $2n - 1$ sign changes.

Fact

The number of noncrossing partitions on n is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}.$$

[Classical; NCPs are interpretation (pp) in Stanley's EC2, Exercise 6.19.]

Fact

The number of **monochromatic** noncrossing partitions on $[2n]$ is

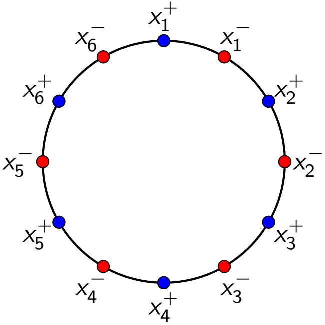
$$\frac{1}{2n+1}\binom{3n}{n}.$$

[Exercise or appendix.]

Bounding the Number of Blocks

Proposition (Block Count Bound)
Every MNP on $[2n]$ has at least $n+1$ blocks.

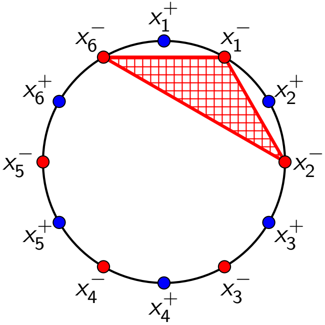
(Proof: Inductive; purely combinatorial/topological.)



Bounding the Number of Blocks

Proposition (Block Count Bound)
Every MNP on $[2n]$ has at least $n+1$ blocks.

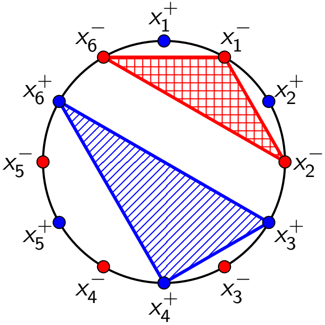
(Proof: Inductive; purely combinatorial/topological.)



Bounding the Number of Blocks

Proposition (Block Count Bound)
Every MNP on $[2n]$ has at least $n+1$ blocks.

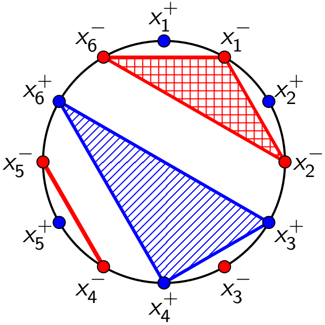
(Proof: Inductive; purely combinatorial/topological.)



Bounding the Number of Blocks

Proposition (Block Count Bound)
Every MNP on $[2n]$ has at least $n+1$ blocks.

(Proof: Inductive; purely combinatorial/topological.)



MNPs and Eigenfunctions

Theorem [HJM '17]

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be $L^2(\mathbb{R})$-eigenvalues of the fractional Schrödinger operator H_α, with eigenfunctions $\phi_1, \ldots, \phi_N$.

Then, each ϕ_n changes sign at most $2(n-1)$ times in $\mathbb{R}$.

Proof. If ϕ_n has $> 2(n-1)$ sign changes, then the Block Count Bond says that $E(\phi_n)$ has at least $n+1$ nodal domains, which contradicts Courant's Theorem.

Periodic Potentials

Suppose that the potential function $V(x)$ for the fractional Schrödinger operator $H_\alpha = (-d^2/dx^2)^{\alpha/2} + V(x)$ is smooth, bounded, and **periodic with period T.**

Theorem [HJM '17]

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be $L^2(\mathbb{R})$-eigenvalues of H_α, with corresponding **periodic** eigenfunctions $\phi_1, \ldots, \phi_N$.

Then, each ϕ_n changes sign at most $2(n-1)$ times **in each period**.

Proof. Essentially the same argument as the nonperiodic case.

The Steklov Problem

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, i.e., an open and connected set whose closure $\overline{\Omega}$ is compact with smooth boundary $\partial\Omega$.

The *Steklov problem* on Ω is

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$.

Fact The solutions to the Steklov problem have eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, accumulating only at $+\infty$

Fact Courant's nodal domain theorem applies: the eigenfunction u_n associated with λ_n has at most n nodal domains in Ω.

Non-Simply Connected Domains

Theorem [HJM '17]

Let Ω be a bounded domain in $\mathbb{R}^2$ of genus g (so that $\partial\Omega$ consists of $g+1$ simple closed curves). Let $f : \overline{\Omega} \to \mathbb{R}$ be a continuous function. Then

$$s \leq 2(n+g-1),$$

where n is the number of nodal domains of f in Ω and s is the number of sign changes of f along $\partial\Omega$.

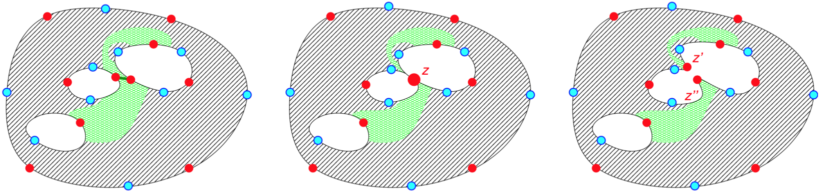
Corollary

The Steklov eigenfunction associated with the Steklov eigenvalue λ_n changes its sign at most $2(n+g-1)$ times on $\partial\Omega$.

Non-Simply Connected Domains

Proof sketch. Induct on genus. Base case $g = 0$ is same as before.

- Find two points x, y on different boundary components that belong to the same nodal domain U.
- Perform "mitosis" on U, which
  - reduces genus by 1
  - increases the number of nodal domains by 1
  - preserves the number of sign changes



Possible Future Directions

These bounds are definitely not sharp!

Do some solutions to some PDE problems admit further analytic constraints that can be translated into combinatorics, thus giving better (possibly even sharp) bounds?

Thanks for listening!

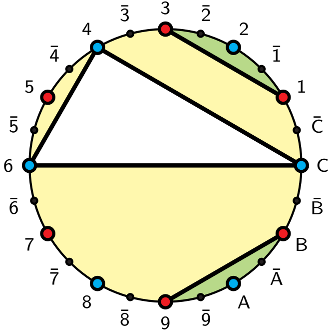
Full paper: Vera Mikyoung Hur, Mathew A. Johnson, and Jeremy L. Martin, Oscillation estimates of eigenfunctions via the combinatorics of noncrossing partitions, *Discrete Analysis* 2017:13, 20 pp.

Let P be an NCP on $[n]$. Its **Kreweras dual** $K(P)$ is as follows.

- Insert a point $\bar{s}$ between each pair of points s and $s+1$.
- Blocks of P dissect the circle into cells that contain the points $\bar{s}$.
- $K(P)$ is then the partition whose blocks are the cells.

$P = \{13,\ 2,\ 46\mathsf{C},\ 5,\ 7,\ 8,\ 9\mathsf{B},\ \mathsf{A}\}$

$K(P) = \{\bar{1}\bar{2},\ \bar{3}\bar{\mathsf{C}},\ \bar{4}\bar{5},\ \bar{6}\bar{7}\bar{8}\bar{\mathsf{B}},\ \bar{9}\bar{\mathsf{A}}\}$



Counting MNPs

- $|K(P)| = n - |P| + 1$
- P monochromatic $\iff K(P)$ 2-*divisible* (all blocks have even cardinality)
- Fact (D. Armstrong): the number of k-divisible NCPs on $[kn]$ is given by the Fuss-Catalan or Raney numbers:

$$\frac{1}{kn+1}\binom{(k+1)n}{n}.$$

Alternative proof of Block Count Bound: A 2-divisible partition of $\{1, \ldots, 2n\}$ can have at most n blocks, so its (monochromatic) Kreweras dual MNP must have at least $n + 1$ blocks