Spanning Trees of Shifted Simplicial Complexes

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The Laplacian of a Graph

G: undirected, loopless, connected graph on vertices $\{1,\ldots,n\}$

$$\ell_{ij} \;=\; \begin{cases} \deg_G(i) & \text{if } i=j, \\ -\#\text{ edges between } i,j & \text{if } i,j \text{ are adjacent}, \\ 0 & \text{otherwise} \end{cases}$$

$L = [\ell_{ij}] =$ **Laplacian matrix** of G

- L is a real symmetric matrix with nonnegative eigenvalues
- $L = MM^{tr}$, where M is the signed incidence matrix of G

The Matrix-Tree Theorem (Kirchhoff, 1847)

Matrix-Tree Theorem, Version I: Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

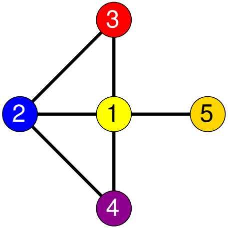
$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$

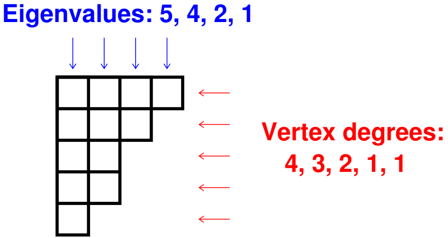
Matrix-Tree Theorem, Version II: Form the *reduced Laplacian* L_i by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i.$$

Example: Threshold Graphs

Theorem (Merris): If G is a *threshold graph*, then the eigenvalues of L are given by the **transpose** of the degree sequence (as a partition).





Simplicial Complexes

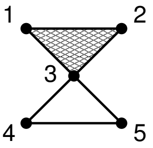
Definition A **simplicial complex on vertex set V** is a family Δ of subsets of V ("faces"), such that

1. $\{v\} \in \Delta$ for every $v \in V$;
2. If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

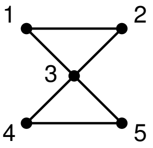
- $\dim F = |F| - 1$; $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$.
- "1-dimensional complex" = "simple graph"

Simplicial Complexes

- **Facets:** maximal faces (under inclusion)
- $\langle F_1, \ldots, F_s \rangle$ = complex with facets $F_1, \ldots, F_s$



<123, 34, 35, 45>



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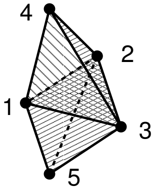
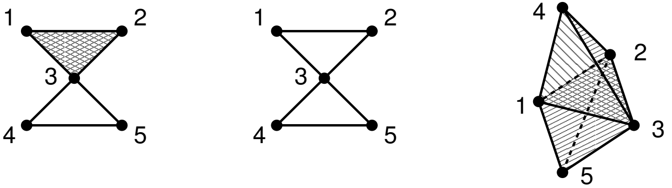


<123, 124, 125, 134,
135, 234, 235>

Simplicial Complexes

- **Pure:** all facets have equal dimension
- **k-skeleton**: $\Delta_{(k)} = \{F \in \Delta \mid \dim F \le k\}$
- **f-vector**: $f_k(\Delta) = \#\{F \in \Delta \mid \dim F = k\}$



not pure	pure	pure
dim = 2	dim = 1	dim = 2
$f(\Delta)$ = (5,6,1)	$f(\Delta)$ = (5,6)	$f(\Delta)$ = (5,9,7)

Simplicial Spanning Trees

Definition Let $\Upsilon \subseteq \Delta$ be pure d-dimensional complexes.
Υ is a **simplicial spanning tree** (SST) of Δ if:

1. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$ ("connected"); and
4. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- Given (0), any two of (1), (2), (3) together imply the third.
- When $d = 1$, we recover the graph-theoretic definition.

Examples of SSTs

Example If $\dim \Delta = 0$, then $\mathrm{SST}(\Delta) = \{\text{vertices of } \Delta\}$.

Example If Δ is $\mathbb{Q}$-acyclic, then $\mathrm{SST}(\Delta) = \{\Delta\}$.

- Includes complexes that are not $\mathbb{Z}$-acyclic, such as $\mathbb{RP}^2$.

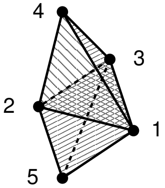
Example If Δ is a simplicial sphere, then

$$\mathrm{SST}(\Delta) = \{\Delta \setminus \{F\} \mid F \text{ a facet of } \Delta\}.$$

- Simplicial spheres are the analogues of cycle graphs.

Simplicial Spanning Trees

Example: Δ = bipyramid with equator.



To construct an SST: Keep the entire 1-skeleton intact, and:

- **either** delete the "equator" (123) and any other triangle,
- **or** delete one northern triangle (124, 134, 234) and one southern triangle (125, 135, 235).

Kalai's Theorem

Theorem [Kalai 1983]

Let $\Delta_{n,d}$ be the d-skeleton of the n-vertex simplex:

$$\Delta_{n,d} = \{F \subset [n] \mid \dim F \leq d\}.$$

Then,

$$\sum_{\Upsilon \in \mathrm{SST}(\Delta_{n,d})} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

- Idea: Calculate det(simplicial Laplacian) in two ways
- Reduces to Cayley's formula for $\Delta_{n,1} = K_n$

The Simplicial Matrix-Tree Theorem

$\Delta = d$-dim'l complex with $\tilde{H}_i(\Delta;\mathbb{Q}) = 0$ for all $i < d$

$$\tau_j(\Delta) = \sum_{\Upsilon \in \mathrm{SST}(\Delta_{(j)})} |\tilde{H}_{j-1}(\Upsilon;\mathbb{Z})|^2$$

Simplicial Matrix-Tree Theorem [D–K–M '07]:

$\tau_d(\Delta) =$  $\times$ (product of nonzero eigenvalues of $L(\Delta)$) / $\tau_{d-1}(\Delta)$

$=$  $\times$ (determinant of reduced Laplacian $L_\Gamma(\Delta)$)

where  and  are correction factors involving simplicial homology; both equal 1 in many cases of interest.

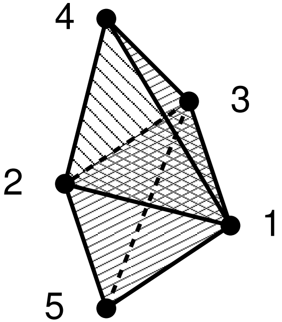
The Simplicial Matrix-Tree Theorem

- The summands $|\tilde{H}_{d-1}(\Upsilon)|^2$ are usually 1 (and are always 1 in the graph case $d = 1$).
- When Δ is a graph, we recover the classical Matrix-Tree Theorem.
- The reduced Laplacian $L_\Gamma(\Delta)$ is obtained by deleting rows and columns corresponding to some $\Gamma \in \mathrm{SST}_{(d-1)}(\Delta)$.
  (When $d = 1$, Γ is a vertex!)

Weighted SST Enumeration

By replacing the simplicial Laplacian (integer entries) with a **weighted version** (with monomial rather than integer entries), we can enumerate simplicial spanning trees more finely.

Example: The Equatorial Bipyramid



Vertices: 1, 2, 3, 4, 5

Edges: All but 45

Facets: 123, 124, 134, 234, 125, 135, 235

$f(\Delta) = (5, 9, 7)$

"Equator": the facet 123

Weighted SST Enumeration in the Equatorial Bipyramid

The full Laplacian L

	12	13	14	15	23	24	25	34	35
12	3	-1	-1	-1	1	1	1	0	0
13	-1	3	-1	-1	-1	0	0	1	1
14	-1	-1	3	0	0	-1	0	-1	0
15	-1	-1	-1	2	0	0	-1	0	-1
23	1	-1	0	0	3	-1	-1	1	1
24	1	0	-1	0	-1	2	0	-1	0
25	1	0	0	-1	-1	0	2	0	-1
34	0	1	-1	0	1	-1	0	2	0
35	0	1	0	-1	1	0	-1	0	2

Weighted SST Enumeration in the Equatorial Bipyramid

The reduced Laplacian L_Γ (delete a 1-dim spanning tree Γ)

	12	13	14	15	**23**	**24**	**25**	**34**	**35**
12	3	-1	-1	-1	1	1	1	0	0
13	-1	3	-1	-1	-1	0	0	1	1
14	-1	-1	3	0	0	-1	0	-1	0
15	-1	-1	-1	2	0	0	-1	0	-1
23	1	-1	0	0	**3**	**-1**	**-1**	**1**	**1**
24	1	0	-1	0	**-1**	**2**	**0**	**-1**	**0**
25	1	0	0	-1	**-1**	**0**	**2**	**0**	**-1**
34	0	1	-1	0	**1**	**-1**	**0**	**2**	**0**
35	0	1	0	-1	**1**	**0**	**-1**	**0**	**2**

Weighted SST Enumeration in the Equatorial Bipyramid

The reduced weighted Laplacian $\hat{L}_\Gamma$

	23	24	25	34	35
23	$x_{123}+x_{234}+x_{235}$	$-x_{234}$	$-x_{235}$	x_{234}	x_{235}
24	$-x_{234}$	$x_{124}+x_{234}$	0	$-x_{234}$	0
25	$-x_{235}$	0	$x_{125}+x_{235}$	0	$-x_{235}$
34	x_{234}	$-x_{234}$	0	$x_{134}+x_{234}$	0
35	x_{235}	0	$-x_{235}$	0	$x_{135}+x_{235}$

where $x_{abc} = x_{1,a}\, x_{2,b}\, x_{3,c}$.

Weighted SST Enumeration in the Equatorial Bipyramid

$c(r, v)$ = number of times vertex v occurs as the r^{th} smallest vertex in a face of Υ

$$\hat{\tau}_2(\Delta) = \sum_{\Upsilon \in \mathrm{SST}(\Delta)} \prod_{F=\{a<b<c\} \in \Upsilon} x_{1,a}\, x_{2,b}\, x_{3,c}$$

$$= \sum_{\Upsilon \in \mathrm{SST}(\Delta)} \prod_{r,v} x_{r,v}^{c(r,v)}$$

$$= x_{11}^3\, x_{22}\, x_{23}^2\, x_{34}^2\, x_{35}^2\, (x_{11}x_{22} + x_{12}x_{22} + x_{12}x_{23})$$
$$\times\ (x_{11}x_{22}x_{33} + x_{12}x_{22}x_{33} + x_{12}x_{23}x_{33} + x_{12}x_{23}x_{34} + x_{12}x_{23}x_{35}).$$

- Erasing all the first subscripts enumerates SSTs by vertex-facet degree sequence.

Shifted Complexes

Definition A simplicial complex Δ on vertices $[n]$ is **shifted** if

- whenever $F \in \Delta$, $i \in F$, $j \notin F$, and $j < i$,
- then $F \setminus \{i\} \cup \{j\} \in \Delta$.

Example If Δ is shifted and $235 \in \Delta$, then Δ must also contain the faces 234, 135, 134, 125, 124, 123.

Shifted Complexes

- Introduced by **Björner and Kalai** (1988) to study f- and Betti numbers
- 1-dimensional shifted complexes = threshold graphs
- Laplacian eigenvalues = transpose of facet-vertex degree sequence (**Duval–Reiner Theorem**, generalizing **Merris' Theorem** for threshold graphs)
- **Conjecture (Duval–Reiner):** For any complex, the Laplacian spectrum is majorized by the degree sequence, with equality iff the complex is shifted
- Graph case: **Grone-Merris Conjecture**, proven by **Bai** (2011)

The Componentwise Partial Order

Define the **componentwise partial order** on $(d+1)$-sets of positive integers

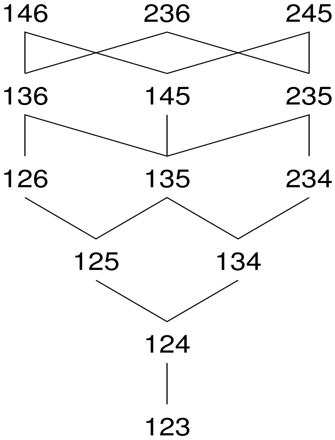
$$A = \{a_1 < a_2 < \cdots < a_{d+1}\},$$
$$B = \{b_1 < b_2 < \cdots < b_{d+1}\}$$

by

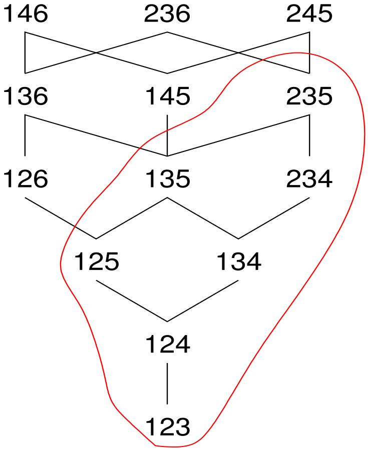
$$A \preceq B \quad \iff \quad a_i \leq b_i \text{ for all } i.$$

- The set of facets of a shifted complex is a *lower order ideal* with respect to $\preceq$.

The Componentwise Partial Order



The Componentwise Partial Order



The Combinatorial Fine Weighting

Let Δ^d be a shifted complex on vertices $[n]$.
For each facet $A = \{a_1 < a_2 < \cdots < a_{d+1}\}$, define

$$x_A = \prod_{i=1}^{d+1} x_{i,a_i} \ .$$

Example If $\Upsilon = \langle 123, 124, 134, 135, 235 \rangle$ is a simplicial spanning tree of Δ, its contribution to $\hat{\tau}_2$ is

$$x_{1,1}^4 x_{1,2} x_{2,2}^2 x_{2,3}^3 x_{3,3} x_{3,4}^2 x_{3,5}^2 \ .$$

The Algebraic Fine Weighting

For faces $A \subset B \in \Delta$ with $\dim A = i - 1$, $\dim B = i$, define

$$X_{AB} = \frac{\uparrow^{d-i} x_B}{\uparrow^{d-i+1} x_A}$$

where $\uparrow x_{i,j} = x_{i+1,j}$.

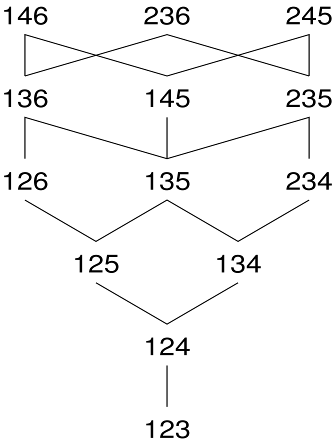
- Weighted boundary maps $\boldsymbol{\partial}$ satisfy $\boldsymbol{\partial}\boldsymbol{\partial} = 0$.
- Laplacian eigenvalues are the same as those for the combinatorial fine weighting, except for denominators.

Critical Pairs

Definition A **critical pair** of a shifted complex Δ^d is an ordered pair (A, B) of $(d+1)$-sets of integers, where

- $A \in \Delta$ and $B \notin \Delta$; and
- B covers A in componentwise order.

Critical Pairs



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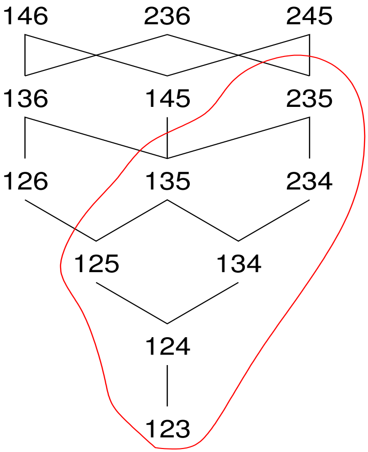
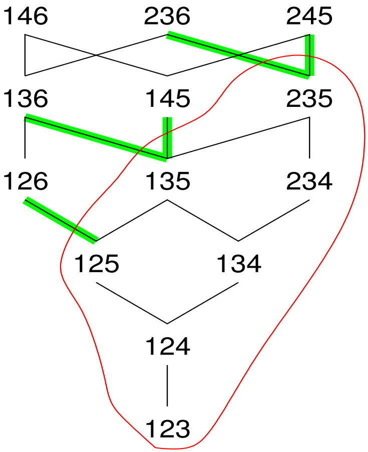


Critical Pairs



The Signature of a Critical Pair

Let (A, B) be a critical pair of a complex Δ:

$$A = \{a_1 < a_2 < \cdots < a_i < \cdots < a_{d+1}\},$$

$$B = A \setminus \{a_i\} \cup \{a_i + 1\}.$$

Definition The **signature** of (A, B) is the ordered pair

$$\big(\{a_1, a_2, \ldots, a_{i-1}\},\ a_i\big).$$

Finely Weighted Laplacian Eigenvalues

Theorem [Duval–Klivans–JLM 2007]

Let Δ^d be a shifted complex.

Then the finely weighted Laplacian eigenvalues of Δ are specified completely by the signatures of critical pairs of Δ.

$$\text{signature } (S, a) \quad \Longrightarrow \quad \text{eigenvalue } \frac{1}{\uparrow X_S} \sum_{j=1}^{a} X_{S \cup j}$$

Examples of Finely Weighted Eigenvalues

- Critical pair (135,145); signature (1,3):

$$\frac{X_{11}X_{21} + X_{11}X_{22} + X_{11}X_{23}}{X_{21}}$$

- Critical pair (235,236); signature (23,5):

$$\frac{X_{11}X_{22}X_{33} + X_{12}X_{22}X_{33} + X_{12}X_{23}X_{33} + X_{12}X_{23}X_{34} + X_{12}X_{23}X_{35}}{X_{22}X_{33}}$$

Sketch of Proof

- Calculate eigenvalues of Δ in terms of eigenvalues of the *deletion* and *link*:

$$\operatorname{del}_1 \Delta = \{F \in \Delta \mid 1 \notin F\},$$
$$\operatorname{link}_1 \Delta = \{F \in \Delta \mid 1 \notin F,\ F \cup \{1\} \in \Delta\}.$$

  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.

Sketch of Proof

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- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$

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- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$

- "Here see ye two recurrences, and lo! they are the same."

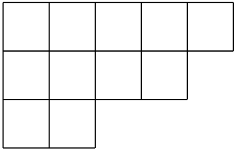
Consequences of the Main Theorem

- Passing to the unweighted version (by setting $x_{i,j} = 1$ for all i, j) recovers the Duval–Reiner theorem.
- Special case $d = 1$: recovers known weighted spanning tree enumerators for threshold graphs (Remmel–Williamson 2002; JLM–Reiner 2003).
- A shifted complex is determined by its set of signatures, so we can "hear the shape of a shifted complex" from its Laplacian spectrum.

Thank you!

Ferrers Graphs

A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.



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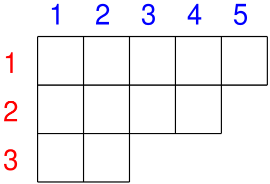

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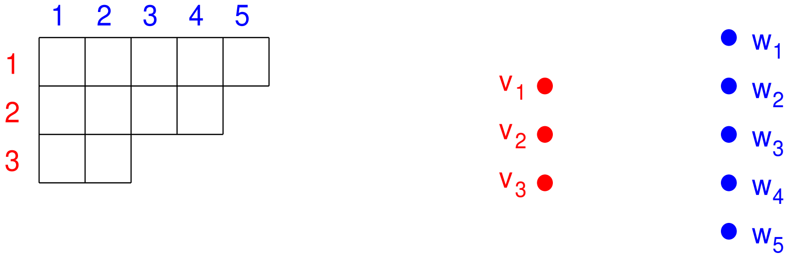

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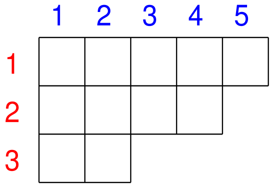

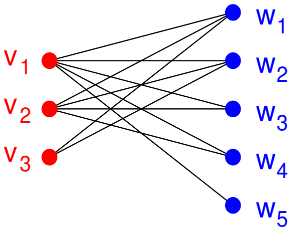

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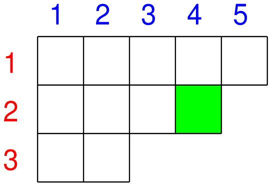

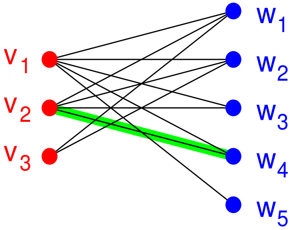

Ferrers Graphs

Ferrers graphs are bipartite analogues of threshold graphs.

- Degree-weighted spanning tree enumerator for Ferrers graphs: Ehrenborg and van Willigenburg (2004)
- Formula can also be derived from our finely weighted spanning tree enumerator for a threshold graph
- Higher-dimensional analogues?

Color-Shifted Complexes

Let Δ be a complex on $V = \bigcup_i V_i$, where

$$V_1 = \{v_{11}, \ldots, v_{1r_1}\}, \; \ldots, \; V_n = \{v_{n1}, \ldots, v_{nr_n}\}.$$

are disjoint vertex sets ("color classes").

Definition Δ is **color-shifted** if

- no face contains more than one vertex of the same color; and
- if $\{v_{1b_1}, \ldots, v_{nb_n}\} \in \Delta$ and $a_i \leq b_i$ for all i, then $\{v_{1a_1}, \ldots, v_{na_n}\} \in \Delta$.

Color-Shifted Complexes

- Color-shifted complexes generalize Ferrers graphs (Ehrenborg–van Willigenburg) and complete colorful complexes (Adin)
- Not in general Laplacian integral. . .
- . . . but they do seem to have nice degree-weighted spanning tree enumerators.

Matroid Complexes

Definition A pure simplicial complex Δ is a **matroid complex** if

- ♢ whenever F, G are facets and $i \in F \setminus G$,
- ♢ there is a vertex $j \in G \setminus F$ such that $F \setminus \{i\} \cup \{j\}$ is a facet.

(Matroid complexes are "maximally egalitarian"; shifted complexes are "maximally hierarchical".)

- Eigenvalues are integers (Kook–Reiner–Stanton 1999), but are harder to describe combinatorially
- Experimentally, degree-weighted spanning tree enumerators seem to have nice factorizations.