A non-partitionable Cohen-Macaulay simplicial complex

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Overview: The Partitionability Conjecture

The focus of this talk is the following conjecture, described in Stanley's Green Book as "a central combinatorial conjecture on Cohen-Macaulay complexes."

Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

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Theorem (DGKM '15+)

The Partitionability Conjecture is **false**. *We construct an explicit counterexample and describe a general method to construct more.*

Partitionable Complexes

X^d = pure simplicial complex of dimension d

A partitioning of X is a decomposition

$$X = \coprod_{\text{facets } F} [R(F), F] \qquad \text{where} \quad [R, F] \stackrel{\text{def}}{=} \{\sigma \mid R \subseteq \sigma \subseteq F\}.$$

- X shellable $\implies$ X partitionable
- X partitionable $\implies$ $h_i(X) = \#\{F : \ |R(F)| = i\} \geq 0$

Cohen-Macaulay and Constructible Complexes

- X^d is Cohen-Macaulay iff its Stanley-Reisner ring (face ring) is Cohen-Macaulay, i.e., $\dim \Bbbk[X] = \operatorname{depth} \Bbbk[X]$. (Combinatorial/topological condition: Reisner's criterion.)
- X^d is constructible iff either it is a simplex, or the union of two constructible d-dimensional complexes whose intersection is constructible of dimension $d-1$.

$$\begin{array}{ccccccc} \text{shellable} & \Longrightarrow & \text{constructible} & \Longrightarrow & \text{CM} & \Longrightarrow & h(X) \geq 0 \\ \Downarrow & & & & & & \\ \text{partitionable} & & & & & & \end{array}$$

Stanley Depth

Definition

Let $S = \Bbbk[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $A \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in S is the vector space

$$\mu \cdot \Bbbk[A] \;=\; \Bbbk\text{-}\operatorname{span}\{\mu\nu \mid \operatorname{supp}(\nu) \subseteq A\}.$$

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of S/I is a family of Stanley spaces

$$\mathcal{D} \;=\; \{\mu_1 \cdot \Bbbk[A_1],\; \ldots,\; \mu_r \cdot \Bbbk[A_r]\}$$

such that

$$S/I \;=\; \bigoplus_{i=1}^{r} \mu_i \cdot \Bbbk[A_i].$$

Stanley Depth

Definition

The Stanley depth of S/I is

$$\operatorname{sdepth} S/I = \max_{\mathcal{D}} \min\{|A_i|\}.$$

where $\mathcal{D}$ runs over all Stanley decompositions of S/I.

For a nice introduction, see M. Pournaki, S. Fakhari, M. Tousi and S. Yassemi, "What is Stanley depth?", Notices AMS 2009

Stanley's Depth Conjecture

Depth Conjecture (Stanley 1982)

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ and $I \subset S$ be any monomial ideal. Then

$$\text{sdepth}\, S/I \geq \text{depth}\, S/I.$$

Theorem (Herzog, Jahan and Tassemi '08)

The Depth Conjecture implies the Partitionability Conjecture

Corollary (DGKM '15+)

The Depth Conjecture is **false**.

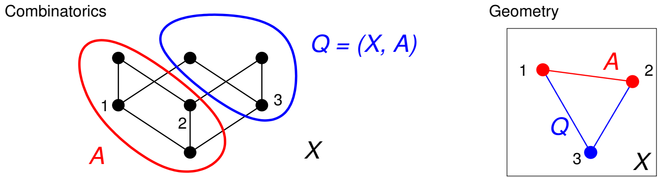
Relative Simplicial Complexes

Definition

A relative simplicial complex Q on vertex set $[n]$ is a convex subset of the Boolean algebra $2^{[n]}$. That is,

$$\sigma, \tau \in Q,\ \sigma \subseteq \rho \subseteq \tau \quad \Longrightarrow \quad \rho \in Q.$$

Every relative complex can be written as $(X, A) = X \setminus A$, where $A \subseteq X$ are simplicial complexes.



Reducing to the Relative Case

$X =$ CM complex $A \subset X$: induced, CM, codim 0 or 1

$Q = (X, A)$ $N > \#$ faces of A

Idea: Construct Ω by gluing N copies of X together along A.

- Ω is CM by Mayer-Vietoris. On the level of face posets,

$$\Omega = Q_1 \cup \cdots \cup Q_N \cup A, \qquad Q_i \cong Q \quad \forall i.$$

- If Ω has a partitioning $\mathcal{P}$, then by pigeonhole $\exists\, i$ such that

$$\exists\, i \in [n] : \quad \forall F \in Q_i : \quad R(F) \notin A.$$

- Therefore, the partitioning of Ω induces a partitioning of Q.

Problem: Find a suitable Q.

Background: Unshellable Balls

Mary Ellen Rudin (1958) constructed a simplicial 3-ball that is not shellable, with f-vector $(1, 14, 66, 94, 41)$ and h-vector $(1, 10, 30)$.

Günter Ziegler (1998) constructed a smaller non-shellable simplicial 3-ball with f-vector $(1, 10, 38, 50, 21)$ and h-vector $(1, 6, 14)$. Its facets are

0**1**23	0**1**2**5**	0237	02**5**6	0267	**1**234	**1**24**9**
12**5**6	**1**26**9**	**1**347	**1**4**5**7	**1**4**5**8	**1**48**9**	**15**6**9**
158**9**	2348	2367	2368	3478	3678	4**5**78

Our Counterexample

Theorem (DGKM 2015+)

Let Z be Ziegler's ball, and let $B = Z|_{0,2,3,4,6,7,8}$.

1. *B is a shellable, hence CM, simplicial 3-ball.*
2. *$Q = (Z, B)$ is not partitionable. Its minimal faces are the three vertices 1, 5, 9.*
3. *Therefore, the simplicial complex obtained by gluing $|B| + 1 = 53$ copies of Z together along B is not partitionable.*

Assertion (2) can be proved by elementary methods.

A Smaller Counterexample

- Let X be the smallest simplicial complex containing Q. Then $Q = (Z, B) = (X, A)$, where

$$f(X) = (1, 10, 31, 36, 14), \qquad f(A) = (1, 7, 11, 5).$$

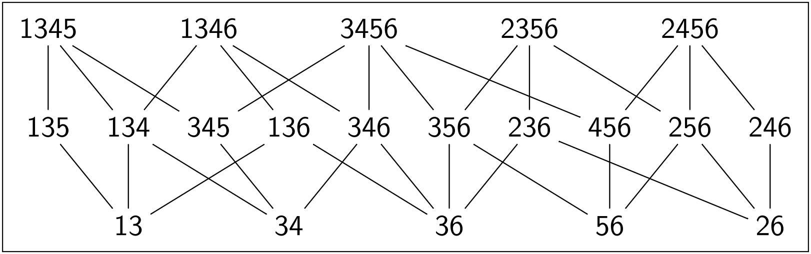
- So a much smaller counterexample can be constructed by gluing together $(1 + 7 + 11 + 5) + 1 = 25$ copies of X along A.
- In fact, gluing three copies of X along A produces a CM nonpartitionable complex Ω, with

$$\mathbf{f(\Omega) = 3f(X) - 2f(A) = (1, 16, 71, 98, 42).}$$

- This is the smallest such complex we know, but there may well be smaller ones.

A Much Smaller Relative Counterexample

There is a much smaller non-partitionable CM relative complex Q' inside Ziegler's ball Z, with face poset



A partitioning of Q' would correspond to a decomposition of this poset into five pairwise-disjoint diamonds. It is not hard to check by hand that no such decomposition exists.

A Much Smaller Relative Counterexample

Construction: $Q' = (X', A')$, where

$$X' = \langle 1589,\ 1489,\ 1458,\ 1457,\ 4578 \rangle\ = Z|_{145789},$$
$$A' = \langle 489,\ 589,\ 578,\ 157 \rangle.$$

- Q' is CM (since X', A' are shellable and $A' \subset \partial X'$)
- $f(Q') = (0, 0, 5, 10, 5)$
- Minimal faces are edges rather than vertices, so Q' cannot be expressed as (X, A) where A is an *induced* subcomplex.
- $\Bbbk[Q']$ is a small counterexample to the Depth Conjecture [computation by Lukas Katthän]

Open Questions

- Is there a smaller counterexample, perhaps in dimension 2?
- What is the "right" strengthening of constructibility that implies partitionability? ("Strongly constructible" complexes, as studied by Hachimori, are partitionable.)
- Is there a better combinatorial interpretation of the *h*-vectors of Cohen-Macaulay complexes? (Duval–Zhang)
- Are all simplicial balls partitionable? (Yes if convex.)
- Does the Partitionability Conjecture still hold for balanced simplicial complexes (as conjectured by Garsia)?
- What are the consequences for Stanley depth? Does sdepth $M \geq$ depth $M - 1$ (as conjectured by Lukas Katthän)?

Thanks for listening!