The uniqueness problem for chromatic symmetric functions of trees

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AMS Western Sectional Meeting
UNLV, April 18, 2015

Colorings and the Chromatic Polynomial

Throughout, $G = (V, E)$ will be a simple graph with $|V| = n$.

A coloring of G is a function $f : V \to \mathbb{N}$ such that

$$vw \in E \implies f(v) \neq f(w).$$

The chromatic function is $\chi_G(k) = \#$ colorings $V \to \{1, \ldots, k\}$.

Well-known facts:

- $\chi_G(k)$ is a polynomial in k.
- If G is a tree then $\chi_G(k) = k(k-1)^{n-1}$.

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- If G is a tree then $\chi_G(k) = k(k-1)^{n-1}$.

- Therefore, the chromatic polynomial of a tree contains no information about it other than the number of vertices.

Symmetric Functions

Definition

A symmetric function is a formal power series $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ that is invariant with respect to all permutations of the variables.

Definition

Let $\lambda = (\lambda_1 \ldots \lambda_\ell) \vdash n$. The monomial symmetric function is

$$m_\lambda = \text{sum of all monomials of the form } x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda^\ell}.$$

Example

$$m_n = \sum_{i=1}^{\infty} x_i^n \qquad m_{11\cdots1} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

The Chromatic Symmetric Function

Let f be a coloring of $G = (V, E)$.

Record the number of times each color is used by a monomial:

$$\mathbf{x}^f \stackrel{\text{def}}{=} \prod_{v \in V} x_{f(v)}.$$

The chromatic symmetric function of G is the formal power series

$$\boxed{X(G) = X_G(x_1, x_2, \ldots) = \sum_{\text{colorings } f} \mathbf{x}^f.}$$

It was introduced by Stanley in 1995.

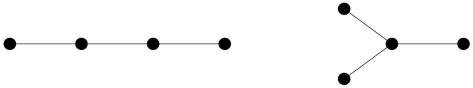
The Chromatic Symmetric Function

- $X(G)$ is a well-defined formal power series, because there are only finitely many colorings with a given "palette".
- $X(G)$ is a symmetric function because permuting the colors does not change whether a coloring is proper.
- $X(G)$ is homogeneous of degree $n = |V(G)|$.
- The chromatic function $\chi_G(k)$ can be recovered from $X(G)$: set $x_1 = \cdots = x_k = 1$, and $x_i = 0$ for all $i > k$.

Examples

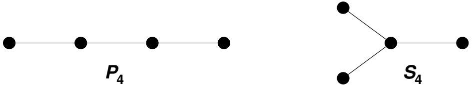
1. $X(K_n) = n! \cdot m_{11\dots1}$ = sum of all squarefree monomials (= elementary symmetric function e_n)
2. $X(\overline{K_n}) = m_1^n$. In general $X(G+H) = X(G)X(H)$.
3. These graphs have the same chromatic symmetric function:

4. These two don't:



Chromatic Symmetric Functions of Trees

For the two trees on 4 vertices...



$$X(P_4) = 24m_{1111} + 6m_{211} + 2m_{22}$$
$$X(S_4) = 24m_{1111} + 6m_{211} + m_{31}$$

Question (Stanley)

Do there exist two non-isomorphic trees with the same CSF?

Power-Sum Symmetric Functions

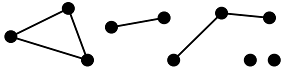
The power-sum symmetric functions are

$$p_n = \sum_{i=1}^{\infty} x_i^n, \qquad p_{(\lambda_1,\ldots,\lambda_\ell)} = \prod_{j=1}^{\ell} p_{\lambda_j}.$$

Fact: The set $\{p_\lambda : \lambda \vdash n\}$ is a vector space basis for $\mathbf{Sym}_n$.

For $S \subseteq E(G)$, let $\mathrm{type}(S)$ = partition whose parts are the sizes of the connected components of the subgraph $(V, S) \subseteq G$.

Example:



type = (3,3,2,1,1)

Power-Sum Coefficients of the CSF

Theorem (Stanley '95)

$$X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\mathrm{type}(S)}.$$

- There is cancellation in Stanley's formula iff G has cycles.

Corollary

If G is a tree and $X(G)$ is written in the basis $\{p_\lambda\}$, then

$$\boxed{\text{coefficient of } p_\lambda \ = \ (-1)^{n-\ell(\lambda)} \#\{A \subseteq E \colon \mathrm{type}(A) = \lambda\}.}$$

The Subtree Polynomial

Definition

The subtree polynomial of a tree T is

$$S_T(q,r) \quad = \quad \sum_{U} q^{|E(U)|}\, r^{|L(U)|} \quad = \quad \sum_{i,j} \sigma_{i,j}(T)\, q^i\, r^j$$

where U ranges over all subtrees of T with at least one edge, and $L(U)$ is the set of leaf edges.

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Theorem (JLM–Matthew Morin–Jennifer Wagner '06)

The numbers $\sigma_{i,j}(T)$ are linear combinations of the $c_\lambda(T)$, with coefficients that depend only on n (not on T).

In particular, $X(T)$ determines $S_T(q,r)$.

Degree and Distance

Corollary

The degree sequence and distance sequence of T are determined by its chromatic symmetric function.

Proof sketch.

Observe that

$$\sigma_{i,i} = \text{ number of } i\text{-edge stars in } T$$
$$\sigma_{i,2} = \text{ number of } i\text{-edge paths in } T$$

Now use inclusion/exclusion to obtain number of vertices with degree i, and number of pairs of vertices at distance i. □

Does This Help Answer Stanley's Question?

This result is enough to prove that some very special classes of trees (e.g., spiders) are determined by their CSFs.

The smallest non-isomorphic trees with the same subtree polynomial have 11 vertices:

A Heroic Computation

Theorem (Keeler Russell, 2013)

Every tree with $n \leq 25$ vertices is determined up to isomorphism by its chromatic symmetric function.

Keeler's proof was entirely a brute-force computation, but with several wrinkles.

- How do you generate all $\approx 10^8$ trees on 25 vertices? (Hint: Do *not* use the Prüfer code followed by isomorphism testing.)
- Trick 1: Classify trees by degree sequence and parallelize
- Trick 2: Compute and compare one coefficient at a time instead of the entire CSF

The Modular Relation

The chromatic symmetric function does not obey a deletion-contraction recurrence, but it does satisfy the modular relation:

Theorem (Guay-Paquet '13+; Orellana–Scott '14)

Suppose that e, e', e'' form a triangle in G. Then

$$X(G) + X(G - e - e') \; = \; X(G - e) + X(G - e').$$

Question

Are there other linear relations between the chromatic symmetric functions of trees (or graphs)?

The theory of combinatorial Hopf algebras may be useful...

Thanks for listening!

Appendix A: Explicit Formula for the Subtree Polynomial

Subtree polynomial:

$$S_T(q,r) \quad = \quad \sum_U q^{|E(U)|} r^{|L(U)|} \quad = \quad \sum_{i,j} \sigma_{i,j}(T) q^i r^j$$

Formula [JLM, Morin, Wagner]:

$$\boxed{\sigma_{i,j} = \sum_{\lambda \vdash n} (-1)^{i+j} \binom{\ell - 1}{\ell - n + i} \sum_{d=1}^{j} \binom{i-d}{j-d} \sum_{k=1}^{\ell} \binom{\lambda_k - 1}{d} c_\lambda(T)}$$

where ℓ = length of λ and $c_\lambda(T)$ = coefficient of p_λ in $X(T)$.

(Are you happy you asked?)