A non-partitionable Cohen-Macaulay simplicial complex

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Kansas State University, February 17, 2016

Preprint: arXiv:1504.04279

Overview: The Partitionability Conjecture

The focus of this talk is the following conjecture, described in Stanley's Green Book as "a central combinatorial conjecture on Cohen-Macaulay complexes."

Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

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Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

Theorem (DGKM '15+)

The Partitionability Conjecture is **false**. *We construct an explicit counterexample and describe a general method to construct more.*

Simplicial Complexes

Let V be a finite set of vertices.
A simplicial complex on V is a family $X \subseteq 2^V$ such that

$$F \in X,\ G \subseteq F \implies G \in X.$$

Equivalently, X is an order ideal in the boolean algebra on V.

- Dimension: $\dim F = |F| - 1$; $\dim X = \max\{\dim F : F \in X\}$.
- Maximal faces of X are called facets.
- X is pure if all facets have the same dimension.
- The complex generated by a list of face(t)s is $\langle F_1, \ldots, F_k \rangle := \bigcup_{i=1}^{k} 2^{F_i}$.

The Stanley-Reisner ring

Let $\mathbb{k}$ be any field, and let X be a simplicial complex of dimension $d-1$ on vertices $V=[n]$.

Associate each $S \subseteq V$ with the monomial $x_S = \prod_{i \in S} x_i$.

The Stanley-Reisner ring of X over $\mathbb{k}$ is

$$\mathbb{k}[X] := \mathbb{k}[x_1, \ldots, x_n] \;/\; \langle x_S \mid S \notin X \rangle.$$

- Graded ring of Krull dimension d
- Algebraic properties of $\mathbb{k}[X] \iff$ combinatorial/topological properties of X

f- and h-vectors

Let X be a simplicial complex of dimension $d-1$.

The f-vector is $f(X) = (f_{-1}, \dots, f_{d-1})$, where

$$f_i = \#\{\text{faces of dimension } i\}.$$

The h-vector $h(X) = (h_0, \dots, h_d)$ is defined by

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}.$$

The h-vector has algebraic significance (it is the numerator of the Hilbert series of $\Bbbk[X]$), and is often positive (e.g., when $\Bbbk[X]$ is Cohen-Macaulay).

What is its combinatorial meaning?

Shellability

A pure simplicial complex X is shellable if its facets can be ordered $F_1, \ldots, F_n$ so that for each k, the set

$$\langle F_1, \ldots, F_k \rangle \setminus \langle F_1, \ldots, F_{k-1} \rangle$$

is an interval $[R_k, F_k]$ in the boolean algebra 2^V.

Proposition

If X is shellable, then $h_i(X) = |\{k \in [n] : \ |R_k| = i\}|$.

But what if $h(X) \geq 0$ but X is not shellable?

Partitionability

Let X be a pure simplicial complex with facets $F_1, \ldots, F_n$.

Definition

A partitioning of X is a decomposition into disjoint Boolean intervals topped by facets:

$$X = \coprod_{i=1}^{n} [R_i, F_i].$$

Note that a partitioning is weaker than a shelling. Nevertheless:

Proposition

If X is partitionable, then $h_i(X) = |\{k \in [n] : \ |R_k| = i\}|$.

Cohen-Macaulay and Constructible Complexes

- X^d is Cohen-Macaulay (CM) iff $\Bbbk[X]$ is CM, i.e., $\dim \Bbbk[X] = \operatorname{depth} \Bbbk[X]$.

- X^d is constructible iff either it is a simplex, or the union of two constructible d-dimensional complexes whose intersection is constructible of dimension $d-1$.

$$\boxed{\begin{array}{ccccccc} \text{shellable} & \Longrightarrow & \text{constructible} & \Longrightarrow & \text{CM} & \Longrightarrow & h(X) \geq 0 \\ \Downarrow & & & & & & \\ \text{partitionable} & & & & & & \end{array}}$$

The Partitionability and Constructibility Conjectures

Theorem (Reisner 1976)

X is Cohen-Macaulay iff for every face $\sigma \in X$,

$$\tilde{H}_i(\mathrm{link}_X(\sigma);\, \mathbb{Z}) = 0 \qquad \forall i < \dim \mathrm{link}_X \sigma.$$

Theorem (Munkres 1984)

The CM condition is topological, i.e., it depends only on the geometric realization $|X|$.

Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

Constructibility Conjecture (Hachimori 2000)

Every constructible simplicial complex is partitionable.

Resolving the Conjectures

Theorem (DGKM 2015+)

The Partitionability and Constructibility Conjectures are false.

We exhibit an explicit simplicial complex Ω that is constructible, hence Cohen-Macaulay, but not partitionable.

Ω is a contractible 3-dimensional complex (but not a ball) with

$$f(\Omega) = (1, 16, 71, 98, 42), \qquad h(\Omega) = (1, 12, 29).$$

Stanley Decompositions and Stanley Depth

Definition

Let $S = \Bbbk[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $A \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in S is the vector space

$$\mu \cdot \Bbbk[A] \;=\; \Bbbk\text{-}\operatorname{span}\{\mu\nu \mid \operatorname{supp}(\nu) \subseteq A\}.$$

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of S/I is a family of Stanley spaces

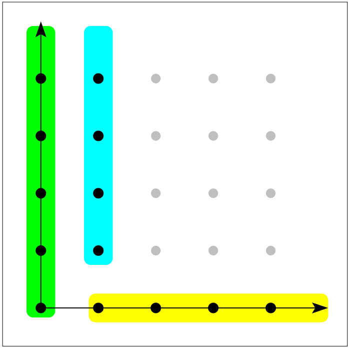
$$\mathcal{D} \;=\; \{\mu_1 \cdot \Bbbk[A_1],\ \ldots,\ \mu_r \cdot \Bbbk[A_r]\}$$

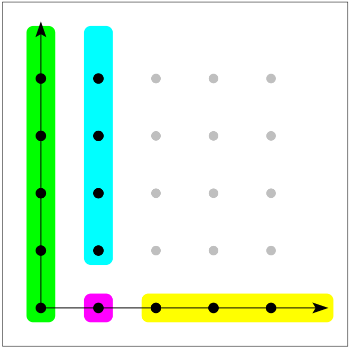
such that

$$S/I \;=\; \bigoplus_{i=1}^{r} \mu_i \cdot \Bbbk[A_i].$$

Stanley Decompositions and Stanley Depth

Two Stanley decompositions of $R = \Bbbk[x, y]/\langle x^2 y\rangle$:





Stanley Decompositions and Stanley Depth

Definition

The Stanley depth of S/I is

$$\operatorname{sdepth} S/I \;=\; \max_{\mathcal{D}} \min\{|A_i|\}.$$

where $\mathcal{D}$ runs over all Stanley decompositions of S/I.

For a nice introduction, see M. Pournaki, S. Fakhari, M. Tousi and S. Yassemi, "What is Stanley depth?", Notices AMS 2009

Stanley's Depth Conjecture

Depth Conjecture (Stanley 1982)
Let $S = \Bbbk[x_1, \ldots, x_n]$ *and* $I \subset S$ *be any monomial ideal. Then*

$$\operatorname{sdepth} S/I \geq \operatorname{depth} S/I.$$

Theorem (Herzog, Jahan and Tassemi '08)
The Depth Conjecture implies the Partitionability Conjecture

Corollary (DGKM '15+)
The Depth Conjecture is **false**.

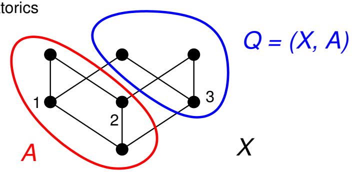
Definition

A relative simplicial complex Q on vertex set V is a convex subset of the Boolean algebra 2^V. That is,

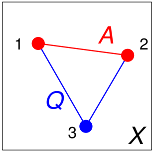
$$F, H \in Q,\ F \subseteq G \subseteq H \quad \Longrightarrow \quad G \in Q.$$

Every relative complex can be written as $(X, A) = X \setminus A$, where $A \subseteq X$ are simplicial complexes.

Combinatorics



Geometry



Relative Simplicial Complexes

Simplicial combinatorics (f- and h-vector, pure, shellable, CM, partitionable, etc.) carries over nicely to the relative setting.

A pure relative simplicial complex Q is Cohen-Macaulay (CM) if a relative version of Reisner's criterion holds, and Q is partitionable if

$$Q = \coprod_{k=1}^{n} [R_k, F_k]$$

where the F_k are the facets of Q.

- Shellable relative complexes are partitionable.
- If $A \subseteq X$ are CM of the same dimension, then so is (X, A).

Reducing to the Relative Case

$X =$ CM complex $A \subset X$: induced, CM, codim 0 or 1
$Q = (X, A)$ $N > \#$ faces of A

Idea: Construct Ω by gluing N copies of X together along A.

- Ω is CM by Mayer-Vietoris. On the level of face posets,

$$\Omega = Q_1 \cup \cdots \cup Q_N \cup A, \qquad Q_i \cong Q \ \ \forall i.$$

- If Ω has a partitioning $\mathcal{P}$, then by pigeonhole $\exists\, i$ such that

$$\exists\, i \in [n] : \quad \forall \text{ facets } F_k \in Q_i : \quad R_k \notin A.$$

- Therefore, the partitioning of Ω induces a partitioning of Q.

Problem: Find a suitable Q.

Background: Unshellable Balls

Mary Ellen Rudin (1958) constructed a simplicial 3-ball that is not shellable, with f-vector $(1, 14, 66, 94, 41)$ and h-vector $(1, 10, 30)$.

Günter Ziegler (1998) constructed a smaller non-shellable simplicial 3-ball with f-vector $(1, 10, 38, 50, 21)$ and h-vector $(1, 6, 14)$. Its facets are

0**1**23	0**1**2**5**	0237	02**5**6	0267	**1**234	**1**24**9**
12**5**6	**1**26**9**	**1**347	**1**4**5**7	**1**4**5**8	**1**48**9**	**15**6**9**
158**9**	2348	2367	2368	3478	3678	4**5**78

Our Counterexample

Theorem (DGKM 2015+)

Let Z be Ziegler's ball, and let $B = Z|_{0,2,3,4,6,7,8}$.

1. *B is a shellable, hence CM, simplicial 3-ball.*
2. *$Q = (Z, B)$ is not partitionable. Its minimal faces are the three vertices 1, 5, 9.*
3. *Therefore, the simplicial complex obtained by gluing $|B| + 1 = 53$ copies of Z together along B is not partitionable.*

Assertion (2) can be proved by elementary methods.

A Smaller Counterexample

- Let X be the smallest simplicial complex containing Q. Then $Q = (Z, B) = (X, A)$, where

$$f(X) = (1, 10, 31, 36, 14), \qquad f(A) = (1, 7, 11, 5).$$

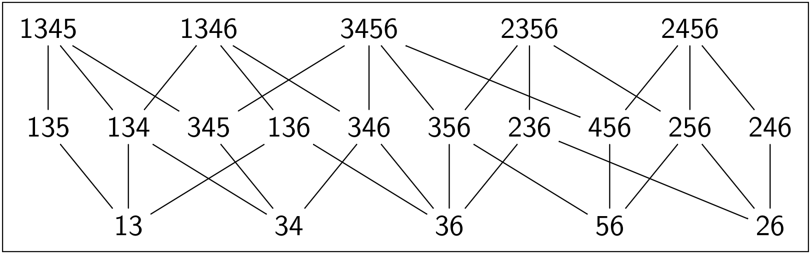
- So a much smaller counterexample can be constructed by gluing together $(1 + 7 + 11 + 5) + 1 = 25$ copies of X along A.
- In fact, gluing three copies of X along A produces a CM nonpartitionable complex Ω, with

$$\boldsymbol{f(\Omega) = 3f(X) - 2f(A) = (1, 16, 71, 98, 42).}$$

- This is the smallest such complex we know, but there may well be smaller ones.

A Much Smaller Relative Counterexample

There is a much smaller non-partitionable CM relative complex Q' inside Ziegler's ball Z, with face poset



A partitioning of Q' would correspond to a decomposition of this poset into five pairwise-disjoint diamonds. It is not hard to check by hand that no such decomposition exists.

A Much Smaller Relative Counterexample

Construction: $Q' = (X', A')$, where

$$X' = \langle 1589,\ 1489,\ 1458,\ 1457,\ 4578 \rangle \ = Z|_{145789},$$
$$A' = \langle 489,\ 589,\ 578,\ 157 \rangle.$$

- Q' is CM (since X', A' are shellable and $A' \subset \partial X'$)
- $f(Q') = (0, 0, 5, 10, 5)$
- Minimal faces are edges rather than vertices, so Q' cannot be expressed as (X, A) where A is an *induced* subcomplex.
- $\mathbb{k}[Q']$ is a small counterexample to the Depth Conjecture [computation by Lukas Katthän]

Open Questions

- Is there a smaller counterexample, perhaps in dimension 2?
- What is the "right" strengthening of constructibility that implies partitionability? ("Strongly constructible" complexes, as studied by Hachimori, are partitionable.)
- Is there a better combinatorial interpretation of the *h*-vectors of Cohen-Macaulay complexes? (Duval–Zhang)
- Are all simplicial balls partitionable? (Yes if convex.)
- Does the Partitionability Conjecture hold for *balanced* simplicial complexes, as conjectured by Garsia? (Bennet Goeckner is working on this.)
- What are the consequences for Stanley depth? Does sdepth $M \geq$ depth $M - 1$ (as conjectured by Lukas Katthän)?

Thanks for listening!