Enumerating cellular colorings, orientations, tensions and flows

Matthias Beck, Felix Breuer (San Francisco State University)
Logan Godkin, Jeremy L. Martin (University of Kansas)

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The chromatic polynomial of a graph

$G = (V, E)$: graph (loops, multiple edges OK) with arbitrary orientation
$n = |V|$, $m = |E|$, $k \in \mathbb{N}$

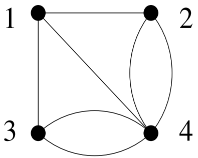
Proper k-coloring: $f : V \to [k]$ with $vw \in E \implies f(v) \neq f(w)$

Chromatic polynomial $\chi_G(k) = \#$ proper k-colorings of G

- $\chi_G(k) =$ polynomial in $k = k^n - mk^{n-1} + \cdots$
- Deletion-contraction: $\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$
- Specialization of Tutte polynomial
- Stanley reciprocity theorem: comb. interp. for $\chi(-k)$

Flows and tensions

Orient G arbitrarily; ∂ = signed incidence/boundary matrix



$$\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

Flow: $(f_e)_{e \in E}$ orthogonal to all rows of ∂
Tension: $(t_e)_{e \in E}$ orthogonal to all flows
Proper coloring: row vector $c = (c_v)_{v \in V}$ with $c\partial$ nowhere-zero

Flows/colorings/tensions can be modular (values in $\mathbb{Z}/k\mathbb{Z}$) or integral (values in $\{-k+1, -k+2, \ldots, k-1\} \subset \mathbb{Z}$)

Flows and tensions

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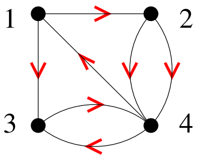

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Modular vs. integral

Modular k-flows/k-tensions

- Flows and tensions form $\mathbb{Z}$-modules [Tutte '47]
- Counted by polynomials in k; specializations of Tutte poly
- Same for any abelian group of cardinality k

Integral k-flows/k-tensions

- Sign vectors correspond to orientations
- Counting functions are polynomials in k [Kochol '02]
- Lattice points in inside-out polytopes [Beck–Zaslavsky '05]
- Reciprocity for flows [Breuer–Sanyal '12]

Cell Complexes

Goal: Extend theory of colorings/cuts/flows from graphs to cell complexes.

X = d-dimensional cell complex
F = facets (d-dimensional faces)
R = ridges (($d-1$)-dimensional faces)

∂ = cellular boundary matrix $\in \mathbb{Z}^{R\times F}$
∂^* = cellular coboundary matrix $\in \mathbb{Z}^{F\times R}$
$\partial_k^* = \partial \otimes \mathbb{Z}/k\mathbb{Z}$

Cellular colorings, flows and tensions

X = pure CW complex F, R = facets, ridges

$K = [-k+1, k-1] \subset \mathbb{Z}$ $\partial = \partial_X \in \mathbb{Z}^{|R|\times|F|}$

Thingamajig	**Definition**	**Enumeration**
Modular coloring	$c \in (\mathbb{Z}_k)^R$ s.t. $c\partial$ nowhere-zero	$\chi^*_X(k)$
Modular flows	$\mathrm{Im}(\partial^*_k)^\perp$	$\varphi^*_X(k)$
Modular tensions	$\mathrm{Im}(\partial^*_k)^{\perp\perp}$	$\tau^*_X(k)$
Integral coloring	$c \in K^R$ s.t. $c\partial$ nowhere-zero	$\chi_X(2k-1)$
Integral flows	$\mathrm{Im}(\partial^*)^\perp \cap K^F$	$\varphi_X(2k-1)$
Integral tensions	$\mathrm{Im}(\partial^*)^{\perp\perp} \cap K^F$	$\tau_X(2k-1)$

Cellular orientations and compatibility

Definition

An orientation of X is a sign vector $\varepsilon \in \{1, -1\}^F$.

An orientation ε and tension/flow $x \in \mathbb{Z}^F$ are compatible if $\varepsilon_f x_f \geq 0$ for every f.

ε is acyclic if it is not compatible with any nonzero flow.

ε is totally cyclic if for every facet f, there is a ε-compatible flow x with $x_f > 0$.

Properties of the modular chromatic function $\chi^*_X(k)$

1. Deletion/contraction for facet/ridge pairs with degree 1
2. Closed formula:

$$\chi^*_X(k) = \sum_{J \subseteq F} (-1)^{|J|} \, |\tilde{H}^d(X_J; \mathbb{Z}_k)| \, k^{n-|J|}$$

3. Quasipolynomial in k; bound on period
4. All ∂_J unimodular $\implies$ polynomial in k, T-G invariant

- Generalizes chromatic polynomial of a graph
- Comparable theorems for tension/flow polynomials (simplicial case: Beck–Kemper)

Integral coloring reciprocity

Theorem

- *Acyclic orientations of* $X \longleftrightarrow$ *regions of hyperplane arrangement* $\mathcal{H}_X$ *with normals = columns of* ∂
- $(-1)^n \chi_X(-2k-1) = \#$ *compatible pairs* (ε, c) *c integral k-coloring,* ε *orientation*
- $|\chi_X(-1)| = \#$ *acyclic orientations of* X

Proof: count lattice points in inside-out polytope $(-1,1)^n \setminus \mathcal{H}_x$; apply Ehrhart-Macdonald reciprocity

(Graph case: Stanley '73, Greene '77)

Integral tension reciprocity

Nowhere-zero integral k-tensions = lattice points in interior of inside-out polytope

$$T = K^F \cap \mathrm{Rowsp}\,\partial \setminus \mathcal{B}$$

where $\mathcal{B}$ = Boolean arrangement of coordinate hyperplanes

Theorem

- *Acyclic orientations of X $\longleftrightarrow$ regions of T*
- $|\tau_X(-2k-1)| = \#$ *compatible pairs* (ε, ψ)*:*
  ψ *integral k-tension,* ε *orientation*
- $|\tau_X(-1)| =$ *number of acyclic orientations*

(Graph case: Chen '10, Dall '08)

Integral flow reciprocity

Nowhere-zero integral k-flows = lattice points in interior of inside-out polytope

$$W = K^F \cap \ker \partial \setminus \mathcal{B}$$

where $\mathcal{B}$ = Boolean arrangement of coordinate hyperplanes

Theorem

- *Totally cyclic orientations of X* $\longleftrightarrow$ *regions of W*
- $|\varphi_X(-2k-1)| = \#$ *compatible pairs* (ε, w)*:*
  w integral k-flow, ε *orientation*
- $|\varphi_X(-1)| =$ *number of totally cyclic orientations*

(Graph case: Beck–Zaslavsky '06)

Modular reciprocity

Modular reciprocity is trickier.

Geometrically: Modular flows/tensions correspond to lattice points in a “periodic inside-out polytope”

Difficult part: How do you associate an orientation (i.e. a sign vector) with a modular flow?

Idea: Breuer–Sanyal '12 (modular flow reciprocity for graphs)
Related work: Chen–Stanley '12

Modular flow reciprocity

Theorem

Let X be a cell complex with no coloops. Then

$$|\varphi^*_X(-k)| = \#\left\{ (\bar{w},\sigma) \colon \begin{array}{l} \bar{w} \textit{ is a } \mathbb{Z}_k\textit{-flow on } X \textit{ and} \\ \sigma : \mathrm{zero}(\bar{w}) \to \{-1,1\} \textit{ extends} \\ \textit{to a totally cyclic orientation} \end{array} \right\}$$

Corollary

$|\varphi^*_X(-1)|$ = *number of totally cyclic orientations*

Modular tension reciprocity

Theorem

Let X be a cell complex with no loops. Then

$$|\tau_X^*(-k)| = \#\left\{ (\bar{t},\sigma) : \begin{array}{l} \bar{t} \text{ is a } \mathbb{Z}_k\text{-tension on } X \text{ and} \\ \sigma : \mathrm{zero}(\bar{t}) \to \{-1,1\} \text{ extends} \\ \text{to an acyclic orientation} \end{array} \right\}$$

Corollary

$|\tau_X^*(-1)| =$ *number of acyclic orientations*

Modular reciprocity: proof sketch

(Idea + graph case: Breuer–Sanyal 2012)

For $k > 0$, interpret $\varphi^*_X(k)$ as sum of Ehrhart functions of disjoint union of components of $(-k, k)^{|F|}$

$\bar{x} \in (\mathbb{Z}_k)^F$ is a flow $\iff$ some (= any) lift $x \in \mathbb{Z}^F$ has $\partial x \in (k\mathbb{Z})^R$

$b \in \mathbb{Z}^R \rightsquigarrow P^\circ(b) = \{w \in (0,1)^F : \partial w = b\}$

$\varphi^*_X(k) = \sum_b \mathrm{Ehr}(P^\circ_b, k)$

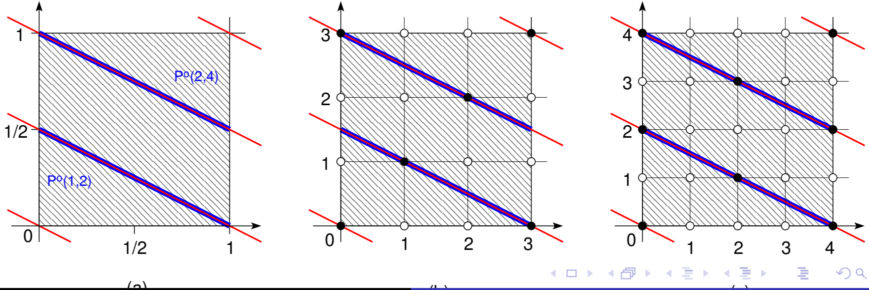
Then apply Ehrhart-Macdonald reciprocity.

Modular reciprocity: proof sketch

Example: $\partial = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$P^\circ(0,0) =$ point $(0,0)$ $\quad P^\circ(1,2) =$ line segment $(1,0)$ to $(0,\frac{1}{2})$
$P^\circ(3,6) =$ point $(1,1)$ $\quad P^\circ(2,4) =$ line segment $(1,\frac{1}{2})$ to $(0,1)$

$\varphi^*_X(k) =$ number of interior lattice points in union of k^{th} dilates
$|\varphi^*_X(-k)| =$ number of lattice points in closed union of k^{th} dilates



Modular reciprocity: proof sketch

- Lattice points on boundaries of $P(b)$'s have coordinates 0 mod k, i.e., somewhere-zero modular flows (which may admit more than one totally cyclic orientation)
- For bijection between these lattice points and $(\bar{w}, \sigma)$, sign $=$ choice of whether to lift 0 mod k to 0 or $k \in \mathbb{Z}$ (requires integral reciprocity!)

Further Directions

1. Is there a **non-TU** cell complex X whose modular chromatic function $\chi^*_X(k)$ is **polynomial**?

2. Kook–Reiner–Stanton ('99): Tutte polynomial of a matroid from convolution of tension and flow polynomials

   Breuer–Sanyal: used KRS to interpret values of Tutte polynomial of a graph at positive integers (a la Reiner '99).

   Generalize to cell complexes whose tension and flow functions are polynomials?

3. **Hopf algebra** point of view: chromatic polynomial = combinatorial Hopf morphism from graphs to polynomials; reciprocity = inversion of characters