Graph Theory and Geometry

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- V is a finite set of *vertices*;
- E is a finite set of *edges*;
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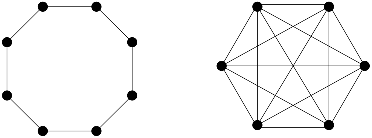
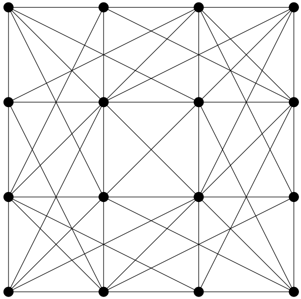


C_8 K_6



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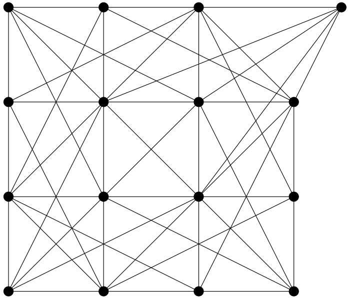

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Why study graphs?

- Real-world applications
  - Combinatorial optimization (routing, scheduling. . .)
  - Computer science (data structures, sorting, searching. . .)
  - Biology (evolutionary descent. . .)
  - Chemistry (molecular structure. . .)
  - Engineering (roads, rigidity. . .)
  - Network models (social networks, the Internet. . .)
- Pure mathematics
  - Combinatorics (ubiquitous!)
  - Discrete dynamical systems (chip-firing game. . .)
  - Algebra (quivers, Cayley graphs. . .)
  - Discrete geometry (polytopes, sphere packing. . .)

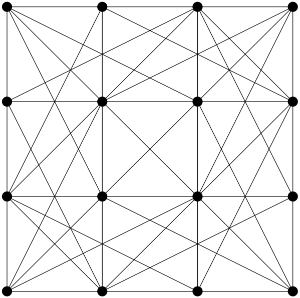
Spanning Trees

Definition A **spanning tree of G** is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Spanning Trees



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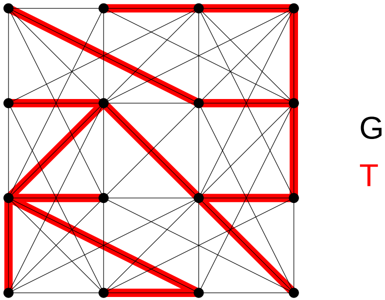

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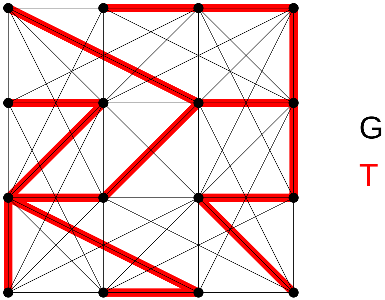

Counting Spanning Trees

$\tau(G)$ = number of spanning trees of G

- $\tau(\text{tree}) = 1$ (trivial)
- $\tau(C_n) = n$ (almost trivial)
- $\tau(K_n) = n^{n-2}$ (Cayley's formula; highly nontrivial!)
- Many other enumeration formulas for "nice" graphs

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Let $e \in E(G)$.

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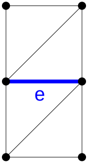

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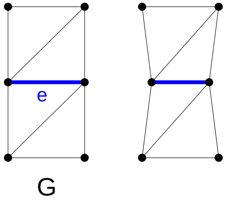

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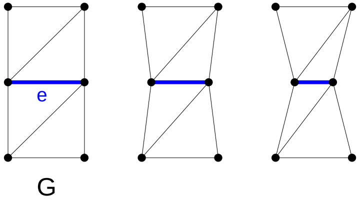

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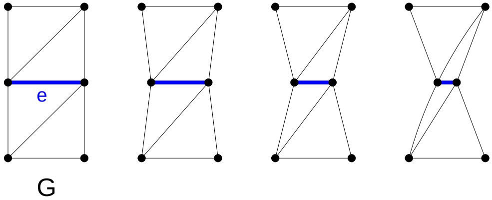

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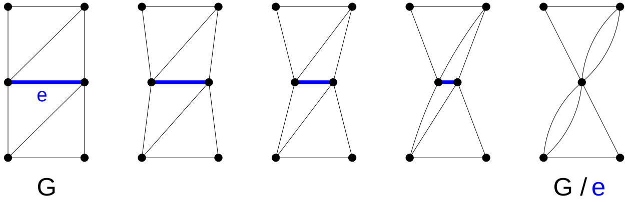

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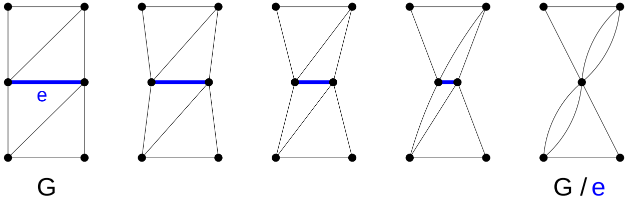

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- Therefore, we can calculate $\tau(G)$ recursively...
- ...but this is computationally inefficient (since $2^{|E|}$ steps must be considered)...
- ...and cannot be used to prove nice enumerative results (like Cayley's formula)

The Matrix-Tree Theorem

$G = (V, E)$: graph with no loops (parallel edges OK)
$V = \{1, 2, \ldots, n\}$

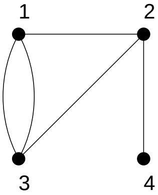
Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -(\#\text{ of edges joining i,j}) & \text{otherwise.} \end{cases}$$

- $\operatorname{rank} L = n - 1.$

The Matrix-Tree Theorem

Example

$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

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The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

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(2) Let $1 \leq i \leq n$. Form the *reduced Laplacian* $\tilde{L}$ by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det \tilde{L} \, .$$

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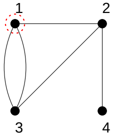
$G =$ 

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Eigenvalues: 0, 1, 4, 5

$(1 \cdot 4 \cdot 5)/4 = 5$

$\det \tilde{L} = 5$

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Theorem $|K(G)| = \tau(G)$.

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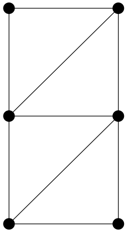
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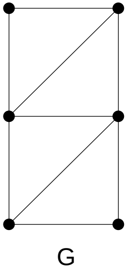

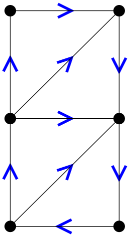

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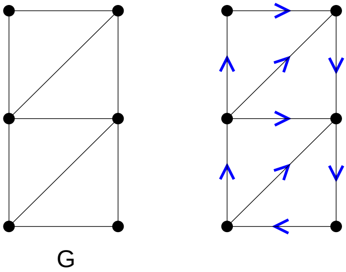

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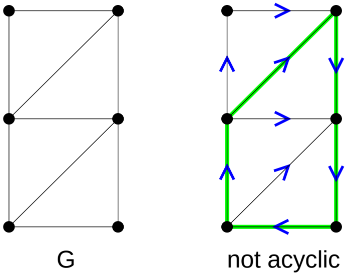

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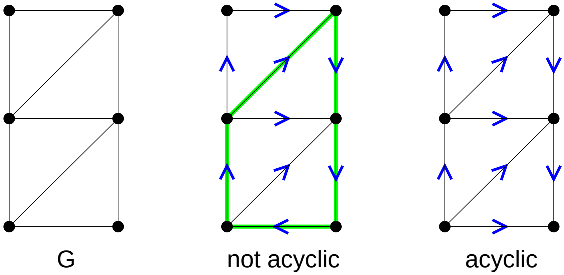

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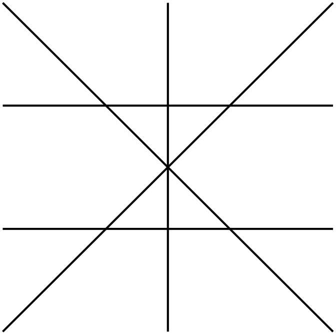
(Fact: Both $\alpha(G)$ and $\tau(G)$, as well as any other invariant satisfying a deletion-contraction recurrence, can be obtained from the *Tutte polynomial* $T_G(x, y)$.)

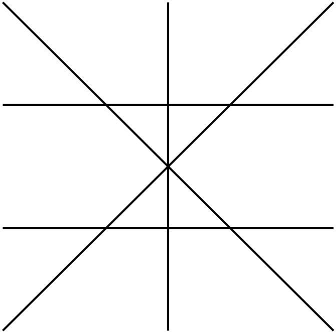
Hyperplane Arrangements

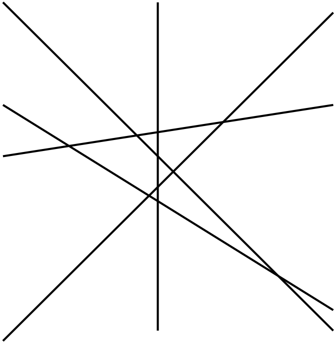
Definition A **hyperplane** H in $\mathbb{R}^n$ is an $(n-1)$-dimensional affine linear subspace.

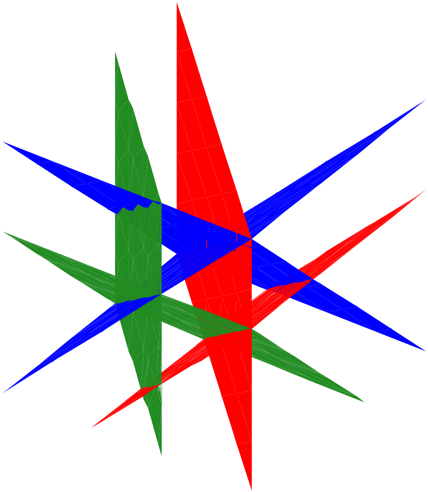
Definition A **hyperplane arrangement** $\mathcal{A} \subset \mathbb{R}^n$ is a finite collection of hyperplanes.

- $n = 1$: points on a line
- $n = 2$: lines on a plane
- $n = 3$: planes in 3-space









Counting Regions

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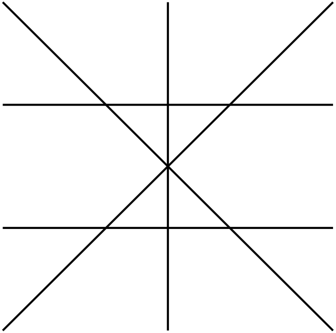

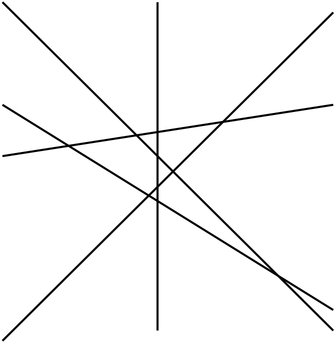

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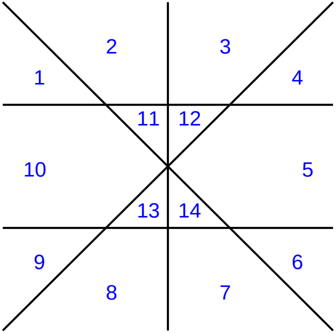
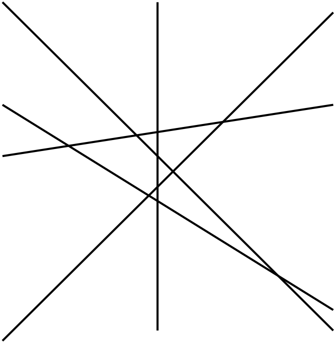


14 regions



16 regions

Counting Regions

Example $\mathcal{A} = n$ lines in $\mathbb{R}^2$

- $2n \leq r(\mathcal{A}) \leq 1 + \binom{n+1}{2}$

Example $\mathcal{A} = n$ coordinate hyperplanes in $\mathbb{R}^n$

- Regions of $\mathcal{A}$ = orthants
- $r(\mathcal{A}) = 2^n$

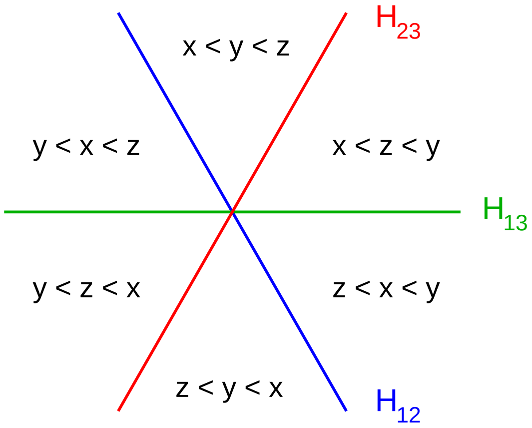
The Braid Arrangement

The *braid arrangement* $Br_n \subset \mathbb{R}^n$ consists of the $\binom{n}{2}$ hyperplanes

$$\begin{aligned} H_{12} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\},\\ H_{13} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\},\\ &\dots\\ H_{n-1,n} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}. \end{aligned}$$

- $\mathbb{R}^n \setminus Br_n = \{\mathbf{x} \in \mathbb{R}^n \mid \text{all } x_i \text{ are distinct}\}$.
- **Problem:** Count the regions of Br_n.

Br_3



Graphic Arrangements

Let $G = (V, E)$ be a simple graph with $V = [n] = \{1, \ldots, n\}$. The *graphic arrangement* $\mathcal{A}_G \subset \mathbb{R}^n$ consists of the hyperplanes

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Theorem There is a bijection between regions of $\mathcal{A}_G$ and acyclic orientations of G. In particular,

$$r(\mathcal{A}_G) = \alpha(G).$$

(When $G = K_n$, the arrangement $\mathcal{A}_G$ is the braid arrangement.)

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Corollary $r(Br_n) = \alpha(K_n) = n!$.

Parking Functions

There are n parking spaces on a one-way street.

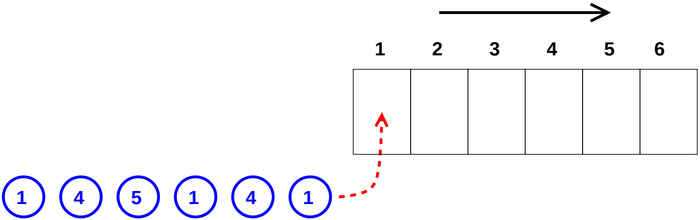
Cars $1, \ldots, n$ want to park in the spaces.

Each car has a preferred spot p_i.

Can all the cars park?

Parking Functions

Example #1: $n = 6$; $(p_1, \ldots, p_6) = (1, 4, 1, 5, 4, 1)$



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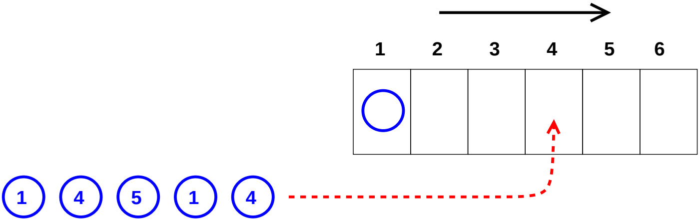

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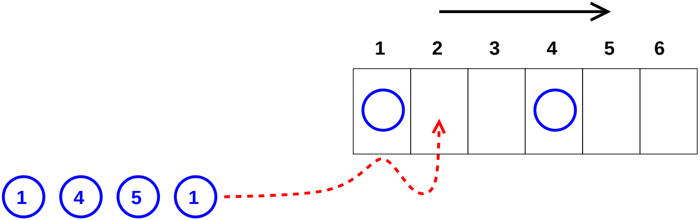

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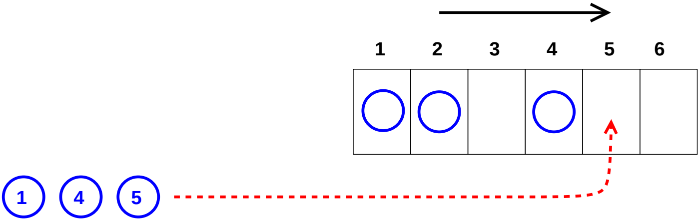

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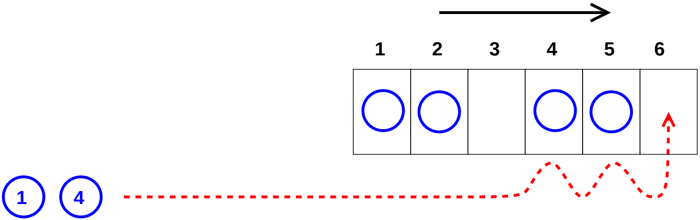
Example #1: $n = 6$; $(p_1, \ldots, p_6) = (1, 4, 1, 5, 4, 1)$



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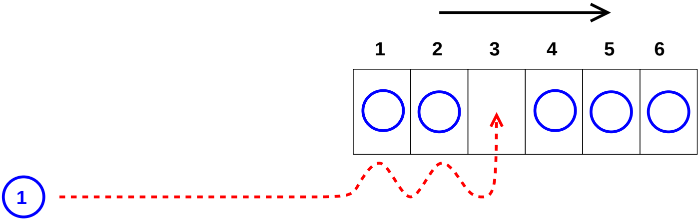

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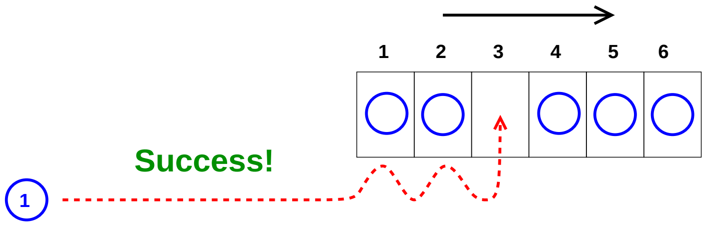

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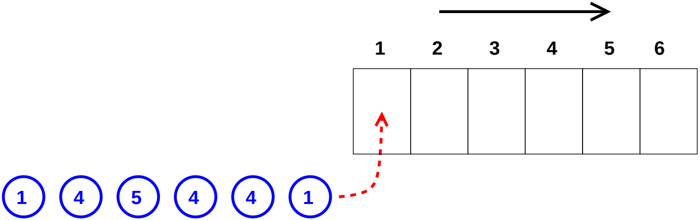

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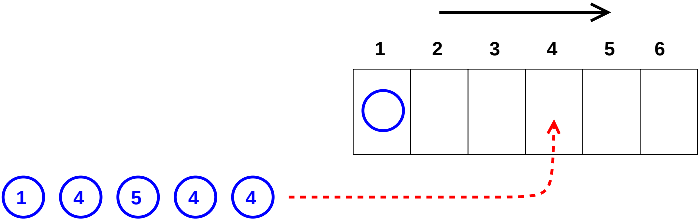

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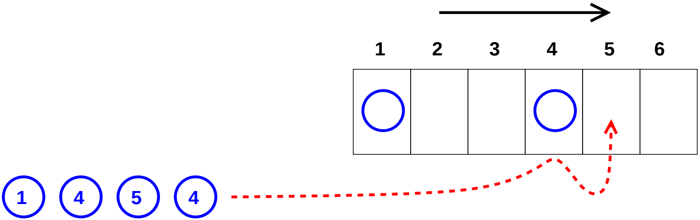

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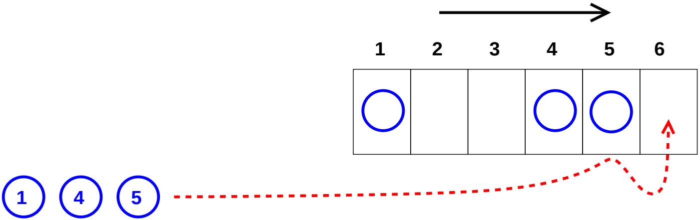

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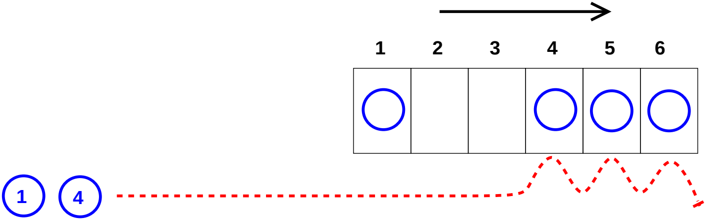

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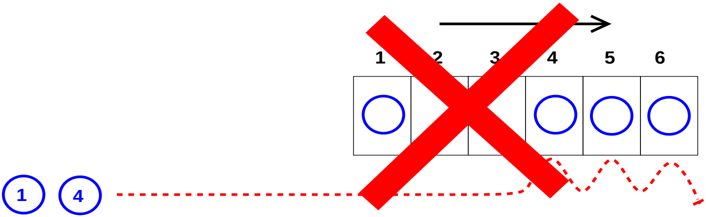

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	121	212	131	213 231
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- In particular, parking functions are invariant up to permutation.

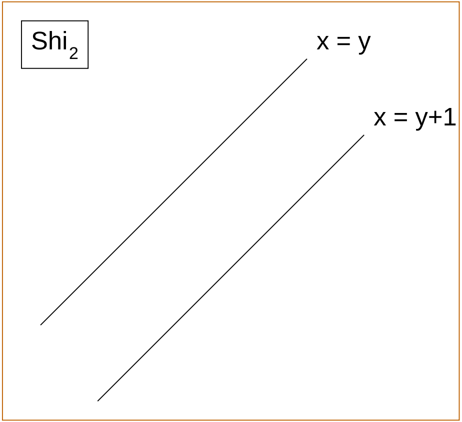
- The number of parking functions of length n is $(n+1)^{n-1}$.

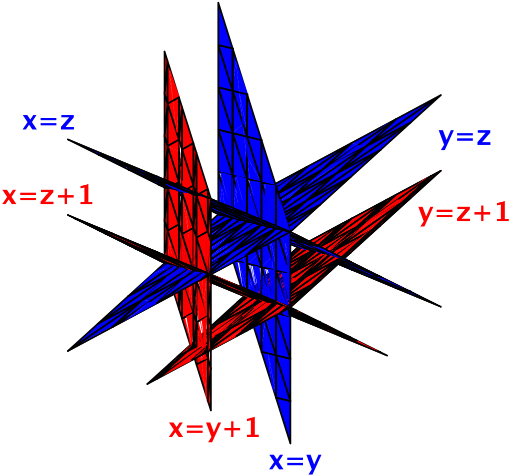
The Shi Arrangement

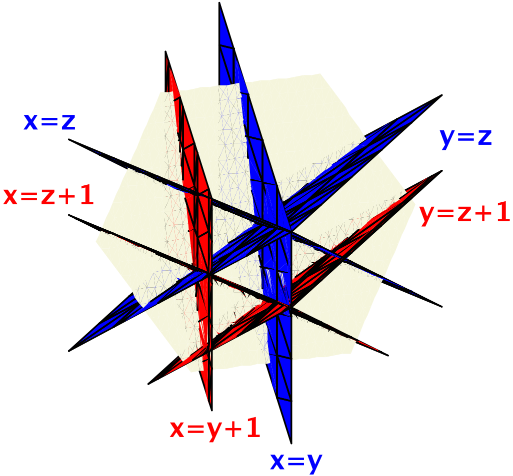
The *Shi arrangement* $Shi_n \subset \mathbb{R}^n$ consists of the $2\binom{n}{2}$ hyperplanes

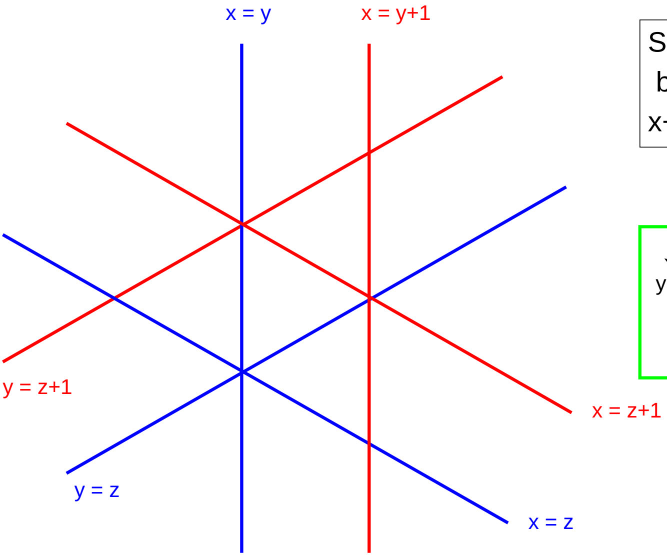
$$\begin{array}{ll} \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2 + 1\}, \\ \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3 + 1\}, \\ \dots & \\ \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n + 1\}. \end{array}$$

The Shi Arrangement

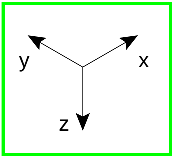



x=z
y=z
x=z+1
y=z+1
x=y+1
x=y


x=z
y=z
x=z+1
y=z+1
x=y+1
x=y



Slice of Shi_3
by plane
$x+y+z = 0$



The Shi Arrangement

Theorem The number of regions in Shi_n is $(n+1)^{n-1}$.

(Many proofs known: Shi, Athanasiadis-Linusson, Stanley . . .)

Score Vectors

Let $\mathbf{x} \in \mathbb{R}^n \setminus Shi_n$. For every $1 \leq i < j \leq n$:

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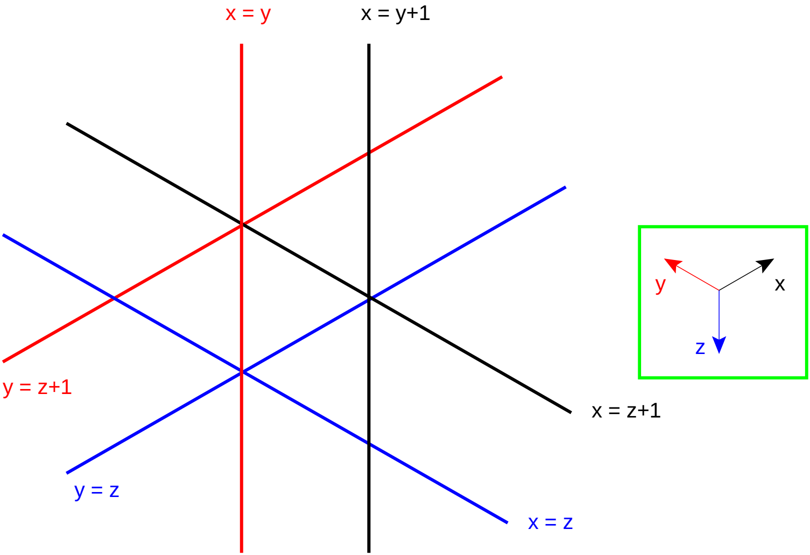
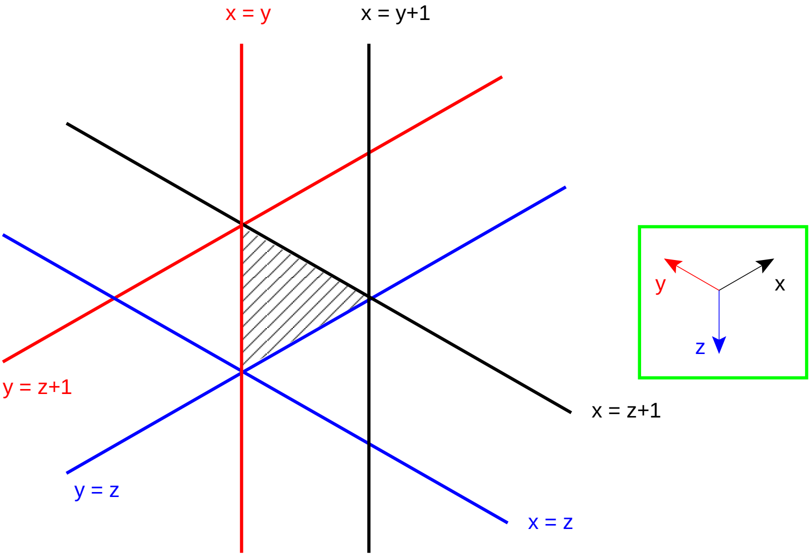
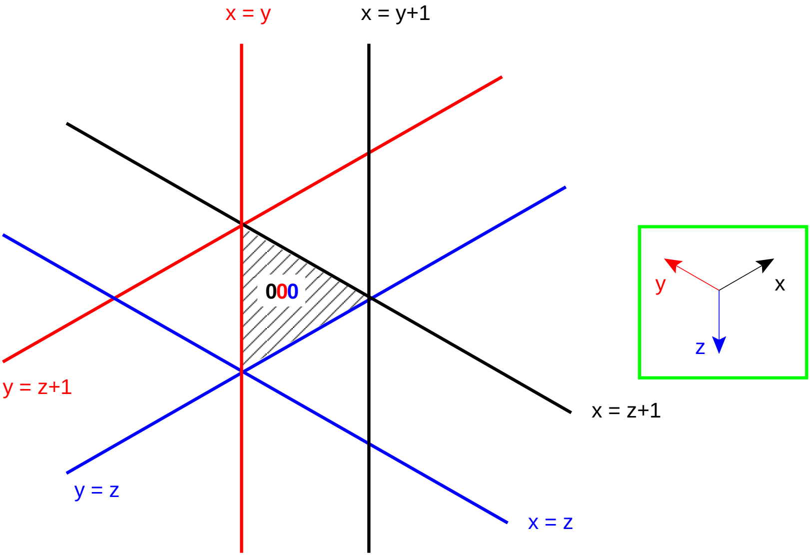
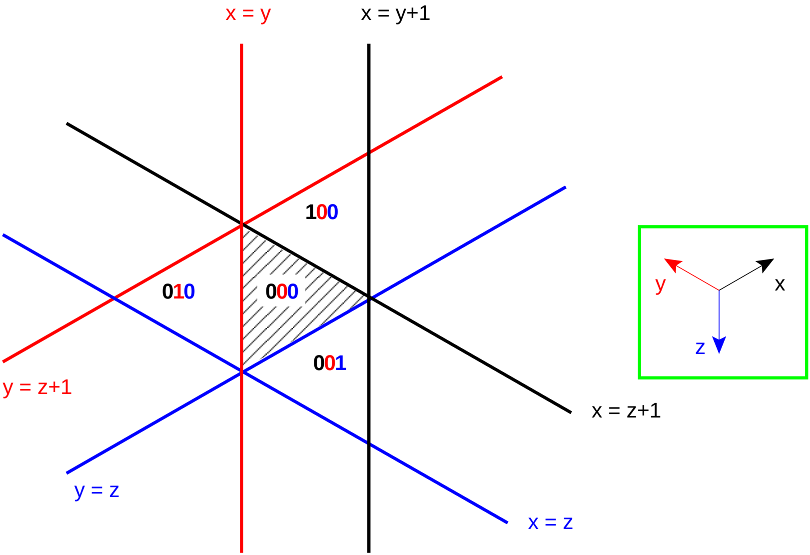
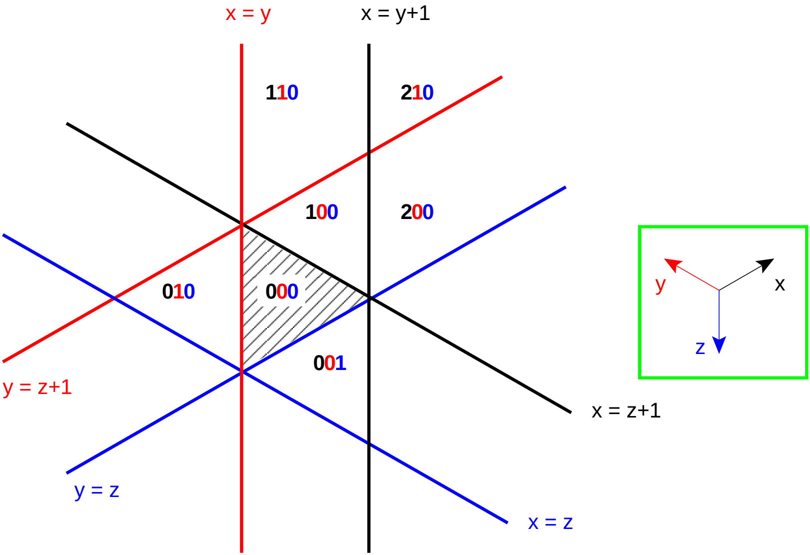
Example The score vector of $\mathbf{x} = (3.142,\ 2.010,\ 2.718)$ is $\mathbf{s} = (1, 0, 1)$.

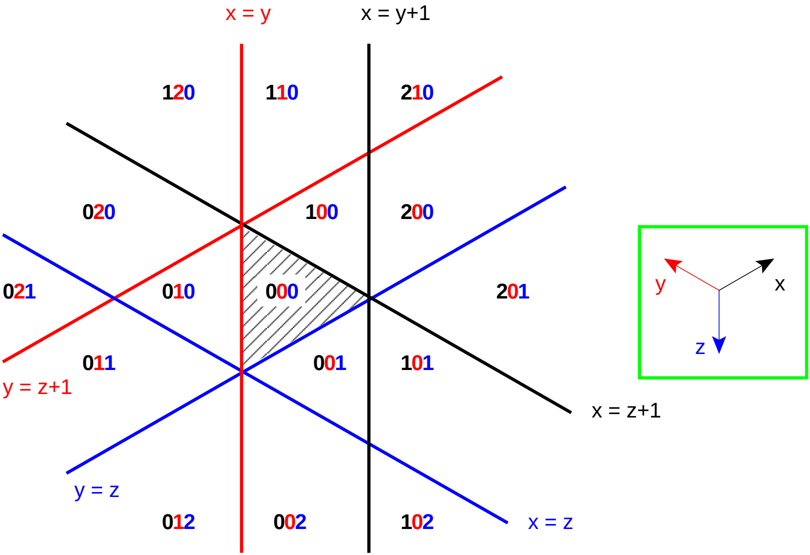

x = y
x = y+1
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
000
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
100
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
110
210
100
200
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
120
110
210
020
100
200
021
010
000
201
011
001
101
y = z+1
x = z+1
y = z
012
002
102
x = z
y
x
z

Score Vectors and Parking Functions

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$$\sum_{\text{regions } R \text{ of } Shi_n} y^{d(R_0,R)} = \sum_{\substack{\text{parking fns} \\ (p_1,\ldots,p_n)}} y^{p_1+\cdots+p_n} = T_{K_{n+1}(1,y)}$$

where d = distance, R_0 = base region.

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where d = distance, R_0 = base region.

Example For $n = 3$: $T_{K_4}(1, y) = 1 + 3y + 6y^2 + 6y^3$.

Simplicial Complexes

Definition An *[abstract] simplicial complex* is a set family

$$\Delta \subseteq 2^{\{1,2,\ldots,n\}}$$

such that

$$\text{if } \sigma \in \Delta \text{ and } \sigma' \subseteq \sigma, \text{ then } \sigma' \in \Delta.$$

The elements of Δ are **simplices**.
The **dimension** of a simplex σ is $|\sigma| - 1$.

- Simplicial complexes are topological spaces, with well-defined homology groups, Euler characteristic, ...

Simplicial Spanning Trees

Definition Let Δ be a simplicial complex of dimension d.

A **simplicial spanning tree** (SST) is a subcomplex $\Upsilon \subset \Delta$ such that:

1. Υ contains all simplices of Δ of dimension $< d$.
2. Υ satisfies appropriate analogues of acyclicity and connectedness (defined in terms of simplicial homology).

Examples of SSTs

- If $\dim \Delta = 1$: SSTs $=$ graph-theoretic spanning trees.

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- If Δ is contractible: it has only one SST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- If Δ is a simplicial sphere: SSTs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any maximal face
  - Simplicial spheres $\approx$ cycle graphs

Kalai's Theorem

Let Δ be the d-skeleton of the n-vertex simplex, i.e.,

$$\Delta = \left\{ F \subseteq \{1, 2, \ldots, n\} \;\middle|\; \dim F \leq d \right\}$$

and let $\mathcal{T}(\Delta)$ denote the set of SSTs of Δ.

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and let $\mathcal{T}(\Delta)$ denote the set of SSTs of Δ.

Theorem [Kalai 1983]

$$\sum_{\Upsilon \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

Kalai's Theorem

- Kalai's theorem reduces to Cayley's formula when $d = 1$ (i.e., when $\Delta = K_n$)
- Anticipated by Bolker (1976), who observed that $n^{\binom{n-2}{d}}$ gave an exact count of trees for small n, d, but failed for $n = 6$, $d = 2$ (the problem is $\mathbb{RP}^2$!)
- Adin (1992): Analogous formula for *complete colorful complexes*, (generalizing known formula for complete bipartite graphs)
- Duval–Klivans–JLM (2007): More general "simplicial matrix-tree theorem" enumerating simplicial spanning trees of many complexes, using combinatorial Laplacians

Open Questions

Does the theory of spanning trees generalize to higher dimension?

- Matrix-Tree Theorem: yes [Duval–Klivans–JLM 2007, extending Bolker 1978, Kalai 1983, Adin 1992]
- Critical group: yes [Duval–Klivans–JLM 2010]
- Acyclic orientations: maybe
- The chip-firing game: doubtful
- Parking functions: also doubtful
- The Shi arrangement: ???