On the Spectra of Simplicial Rook Graphs

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The Adjacency and Laplacian Matrices of a Graph

Let $G = (V, E)$ be a simple graph.

Adjacency matrix $A(G)$: rows and columns indexed by $V(G)$; with 1s for edges, 0s for non-edges

Laplacian matrix $L(G)$: $D - A$, where D = diagonal matrix of vertex degrees

- Eigenvalues of A and L are invariants that encode connectivity, number of spanning trees, ...
- If G is regular (all vertices have the same degree), then A, L have same eigenspaces

Simplicial Rook Graphs

Definition

For $d, n \in \mathbb{N}$, consider the dilated simplex

$$\Delta = \Delta_n^{d-1} = \{\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d \colon \sum_{i=1}^{d} v_i = n\}.$$

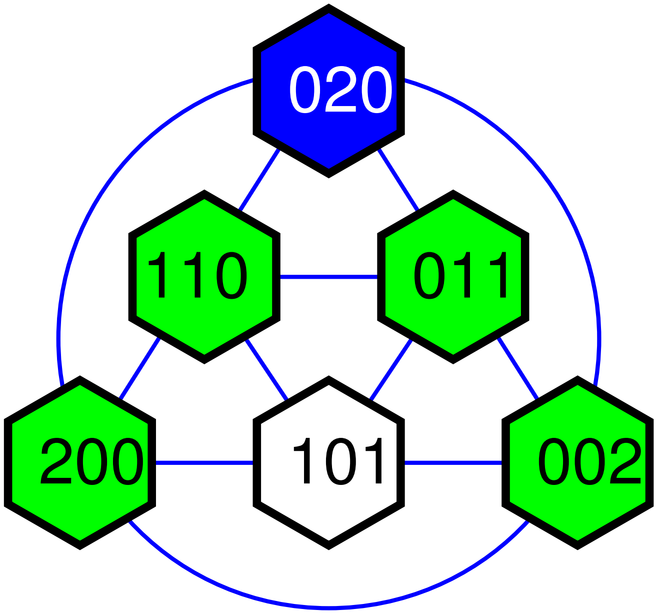
The **simplicial rook graph** $SR(d, n)$ is the graph with vertices

$$V(d, n) = \Delta_n^{d-1} \cap \mathbb{N}^d$$

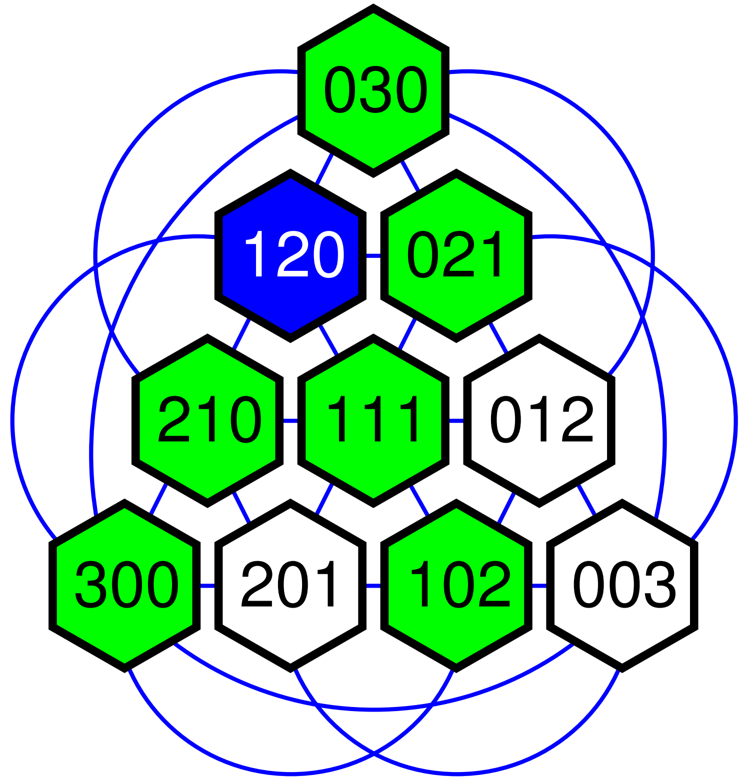
with edges $\{\mathbf{vw} \colon \mathbf{v}, \mathbf{w}$ differ in exactly 2 coordinates$\}$.

- $|V(d, n)| = \binom{n+d-1}{d-1}$
- $SR(d, n)$ is regular of degree $\delta = (d-1)n$
- $SR(2, n) = K_{n+1}$

Example: $SR(2,3)$ and $SR(3,3)$



SR(2,3)
degree 4
(= octahedron)



SR(3,3)
degree 6

The Spectrum of $SR(3, n)$

Theorem (JLM–JDW 2012)

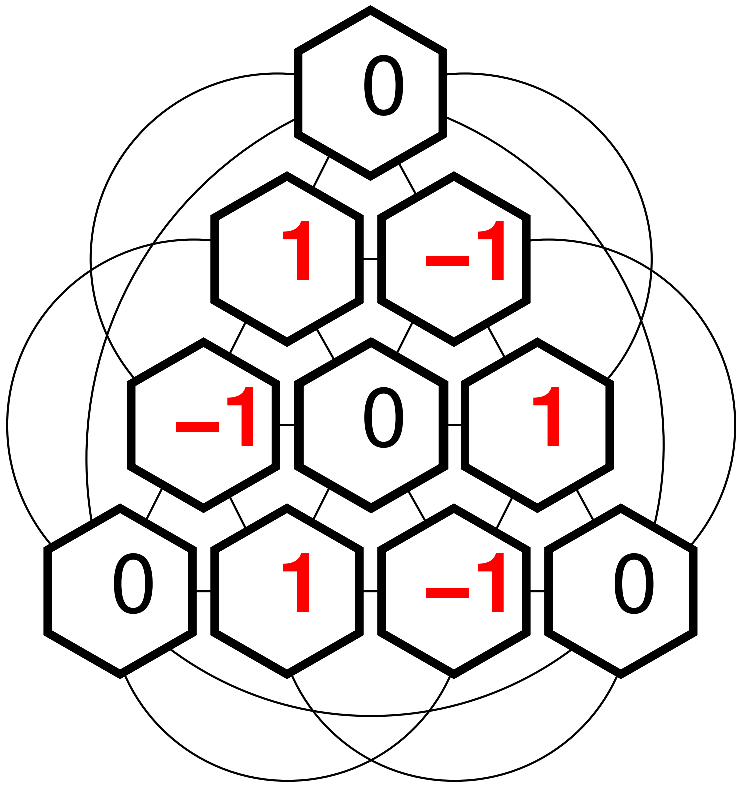
The eigenvalues of the adjacency matrix A of $SR(3, n)$ are:

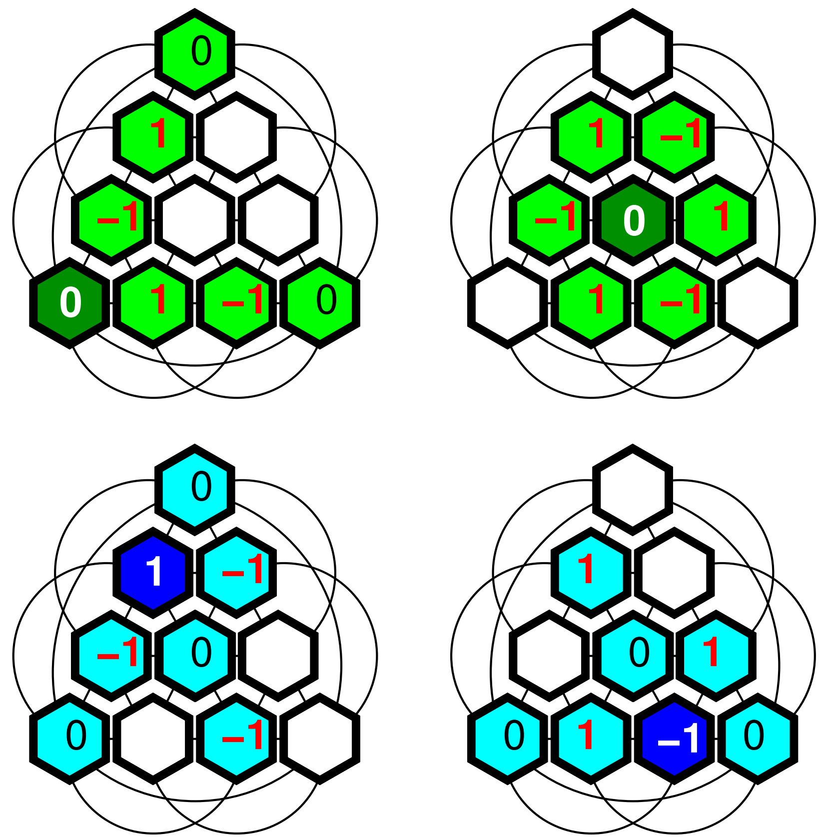
$\mathbf{n = 2m + 1}$ **odd**		$\mathbf{n = 2m}$ **even**	
Eigenvalue	**Multiplicity**	**Eigenvalue**	**Multiplicity**
-3	$\binom{2m}{2}$	-3	$\binom{2m-1}{2}$
$-2, \ldots, m-3$	3	$-2, \ldots, m-4$	3
$m-1$	2	$m-3$	2
$m, \ldots, n-2$	3	$m-1, \ldots, n-2$	3
$2n$	1	$2n$	1

Note: A acts on $\mathbb{R}V$ by $A[\mathbf{v}] = \sum_{\text{neighbors } \mathbf{w} \text{ of } \mathbf{v}} [\mathbf{w}]$.

Hex Vectors

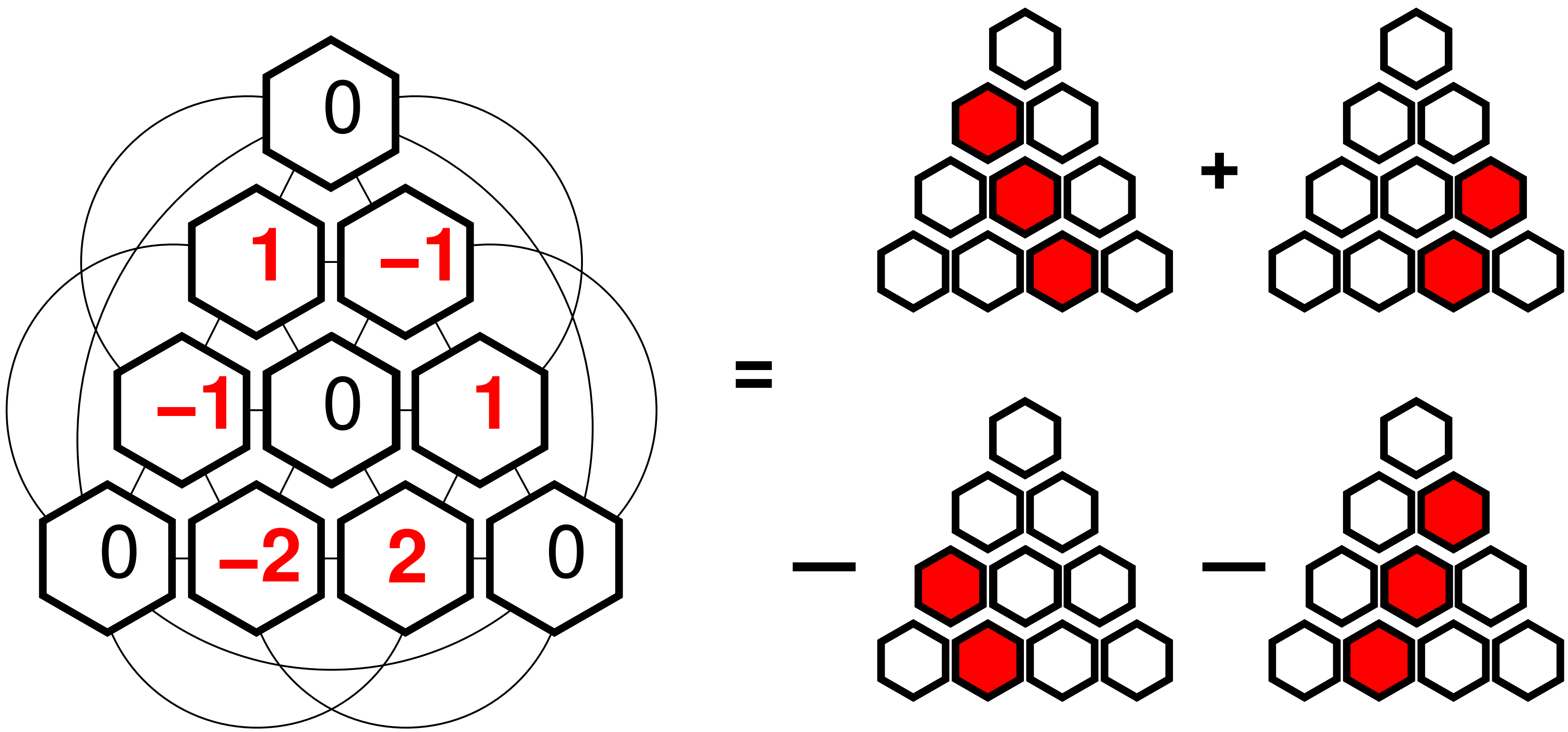
When **v** is an "interior" vertex ($v_i > 0$ for all i), the hexagon centered at **v** gives rise to an eigenvector with eigenvalue -3.





Eigenvectors of $A(3, n)$

- Number of possible centers for a hexagon vector = number of interior vertices in $V(3, n) = \binom{n-1}{2}$.
- The hexagon vectors are all linearly independent.
- The other $\binom{n+2}{2} - \binom{n-1}{2} = 3n$ eigenvectors are sums of characteristic vectors of lattice lines. For example:



Simplicial Rook Graphs in Arbitrary Dimension

Conjecture

The graph $SR(d, n)$ is integral for all d and n.

- Experimental evidence: verified by direct calculation for

$$d = 4, n \leq 25 \qquad d = 5, n \leq 15$$

$$d = 6, n \leq 10 \qquad d = 7, n \leq 7$$

- Partial results: complete geometric description of (asymptotically) largest eigenspace

Permutohedron Vectors

Definition

A **lattice permutohedron** in $\mathbb{R}^d$ is a set of $d!$ points of the form

$$\mathrm{Per}(\mathbf{p}) = \{\mathbf{p} + \mathbf{w} \colon \mathbf{w} \in \mathfrak{S}_d\}$$

where $\mathbf{p} \in \mathbb{Z}^d$ and $\mathfrak{S}_d$ is the set of permutations of $(1, 2, \ldots, d)$.

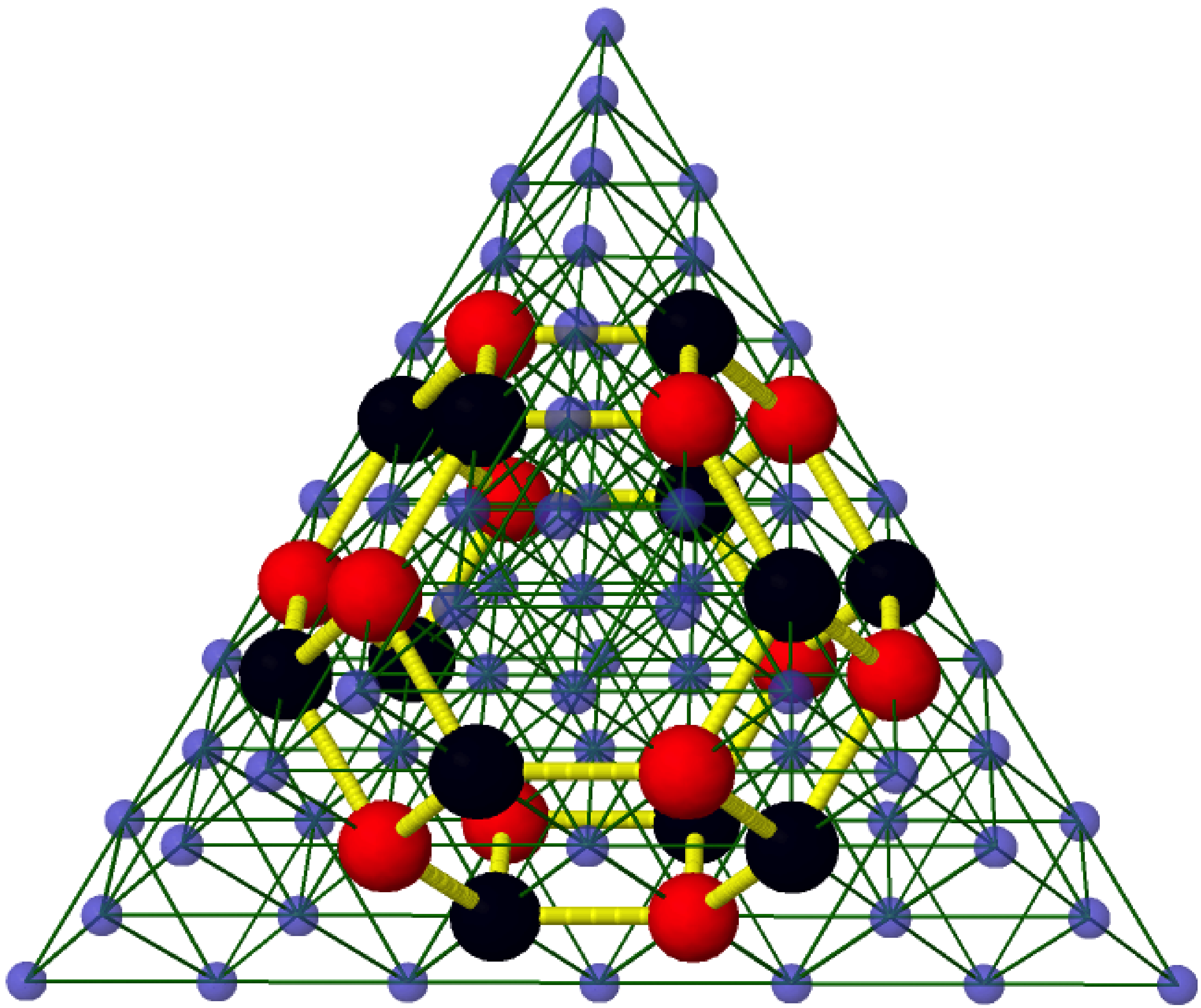
Theorem

If $\mathrm{Per}(\mathbf{p}) \subseteq V(d, n)$, *then the vector*

$$H_{\mathbf{p}} = \sum_{\mathbf{w} \in \mathfrak{S}_d} sign(\mathbf{w})[\mathbf{p} + \mathbf{w}]$$

is an eigenvector of A *with eigenvalue* $-\binom{d}{2}$.

A Lattice Permutohedron in $SR(4, 6)$



Permutohedron Eigenvectors

- The vectors $H_{\mathbf{p}}$ are linearly independent.

- Permutohedron vectors account for "most" eigenvectors:

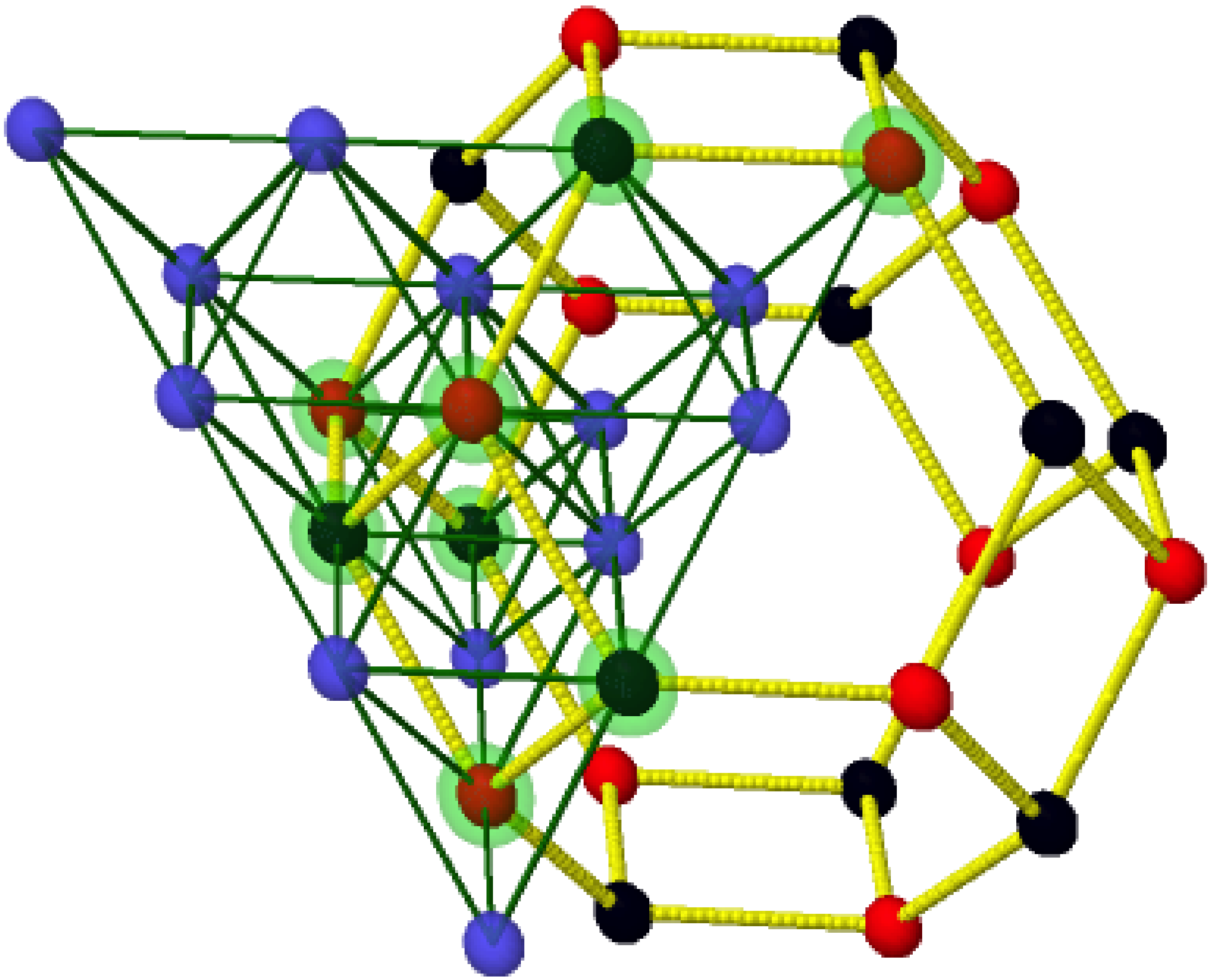
$$\frac{\#\{\mathbf{p}\colon\ \mathrm{Per}(\mathbf{p}) \subset V(d,n)\}}{|V(d,n)|} = \frac{\binom{n-\binom{d-1}{2}}{d-1}}{\binom{n+d-1}{d-1}} \to 1 \quad \text{as } n \to \infty.$$

- $-\binom{d}{2}$ is the smallest eigenvalue of $SR(d,n)$.

- In order for Δ_n^{d-1} to contain any lattice permutohedra, we must have $n \geq \binom{d}{2}$.

The Case $n < \binom{d}{2}$

When $n < \binom{d}{2}$, the simplex Δ_n^{d-1} contains no lattice permutohedra. On the other hand, characteristic vectors of **partial permutohedra** are eigenvectors with eigenvalue $-n$.



The Case $n < \binom{d}{2}$

Theorem

If $n \leq \binom{d}{2}$, then every permutation $\pi \in \mathfrak{S}_d$ with n inversions gives rise to an eigenvector F_π of $A(SR(d,n))$ with eigenvalue $-n$. Moreover, these eigenvectors are linearly independent.

- The number of F_π is the Mahonian number $M(d,n) =$ coefficient of q^n in

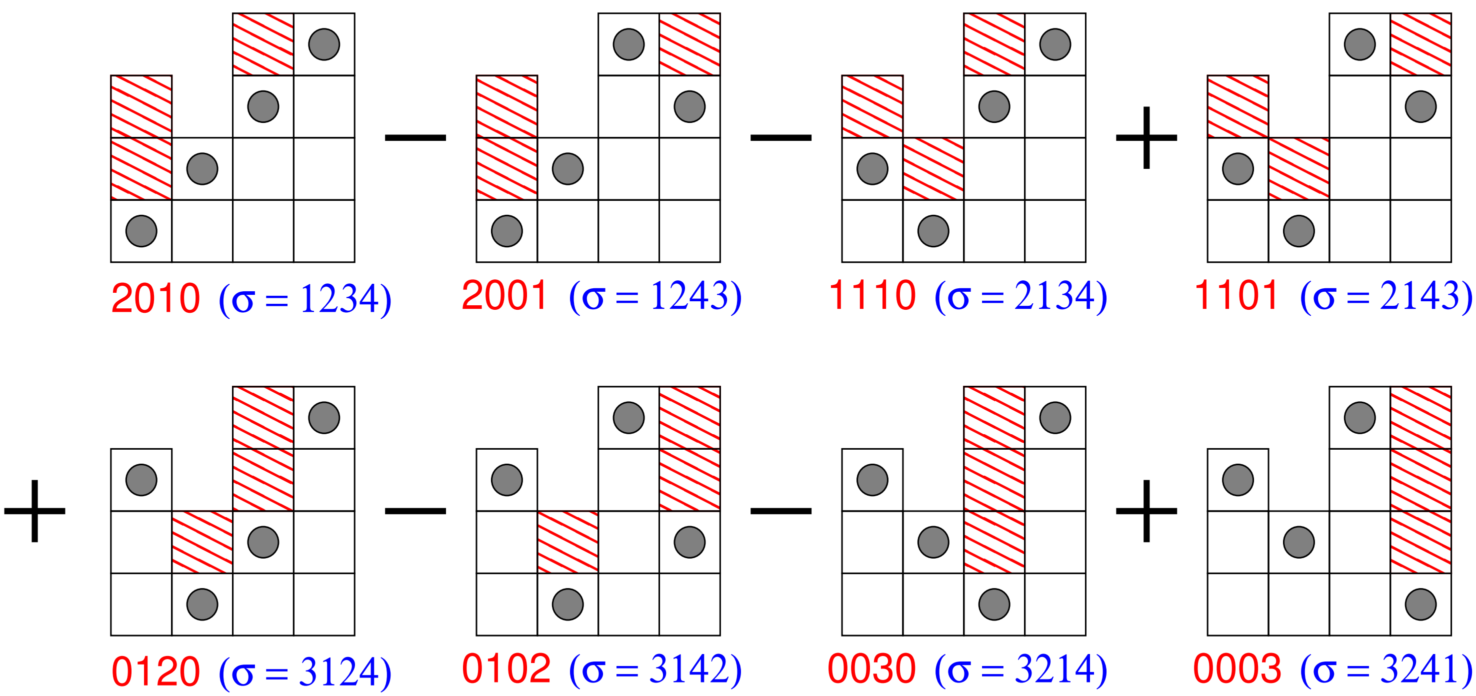
$$(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{d-1}).$$

- The F_π appear to be a complete list of lowest-weight eigenvectors.
- Construction of F_π uses (ordinary, non-simplicial) rook theory.

Constructing an Eigenvector F_π

Example: $n = 3,\ d = 4,\ \pi = 3142 \in \mathfrak{S}_d$

- Let $\mathbf{a} = (a_i)_{i=1}^{d}$, where $a_i = \#\{j > i : \pi(j) < \pi(i)\}$. Here, $\mathbf{a} = (2, 0, 1, 0)$.
- $F_\pi = \sum_\sigma \operatorname{sign}(\sigma)[\mathbf{b} - \sigma]$, where σ runs over all rook placements on the skyline board $\mathbf{b} = \mathbf{a} + (1, \ldots, d)$.



Open Problems

- Prove that $A(d,n)$ (equivalently, $L(d,n)$) has integral spectrum for all d,n.

- Prove that the induced subgraphs

$$SR(d,n)|_{V(d,n)\cap \mathrm{Per}(\mathbf{p})}$$

  also appear to be Laplacian integral for all $d,n,\mathbf{p}$. (Verified for $d \leq 6$.)

- Is $A(d,n)$ determined up to isomorphism by its spectrum? (We don't know.)