Graph Theory and Discrete Geometry

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Graphs

A **graph** is a pair $G = (V, E)$, where

- V is a finite set of *vertices*;
- E is a finite set of *edges*;
- Each edge connects two vertices called its *endpoints*.

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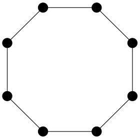
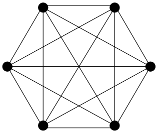
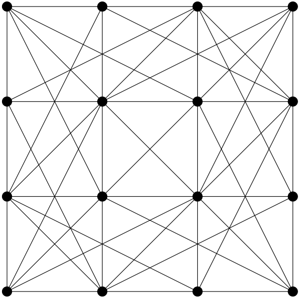


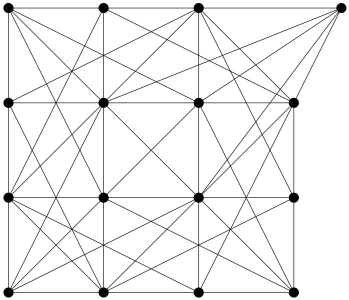
C_8



K_6



G



G

Why study graphs?

- Real-world applications
  - Combinatorial optimization (routing, scheduling. . .)
  - Computer science (data structures, sorting, searching. . .)
  - Biology (evolutionary descent. . .)
  - Chemistry (molecular structure. . .)
  - Engineering (roads, electrical circuits, rigidity. . .)
  - Network models (the Internet, Facebook!. . .)
- Pure mathematics
  - Combinatorics (ubiquitous!)
  - Discrete dynamical systems (chip-firing game. . .)
  - Abstract algebra. . .)
  - Discrete geometry (polytopes, sphere packing. . .)

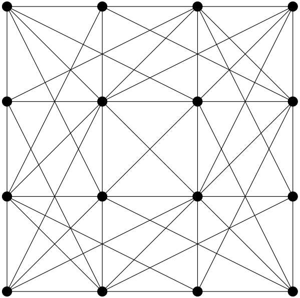
Spanning Trees

Definition A **spanning tree of G** is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Spanning Trees



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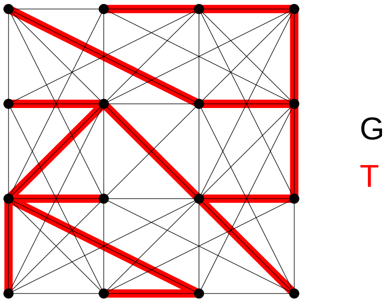

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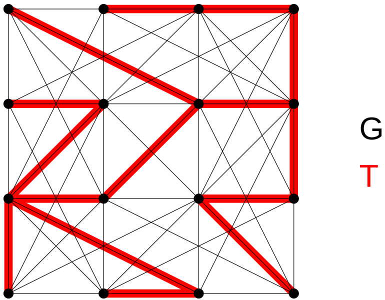

Counting Spanning Trees

Definition $\tau(G)$ = number of spanning trees of G

(Think of $\tau(G)$ as a rough measure of the complexity of G.)

- $\tau(\text{tree}) = 1$ (trivial)
- $\tau(C_n) = n$ (almost trivial)
- $\tau(K_n) = n^{n-2}$ (Cayley's formula; highly nontrivial!)
- Many other enumeration formulas for "nice" graphs

Deletion and Contraction

Let $e \in E(G)$.

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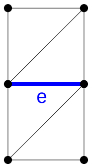

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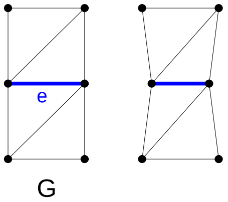

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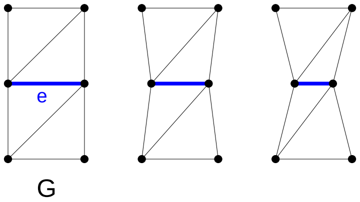

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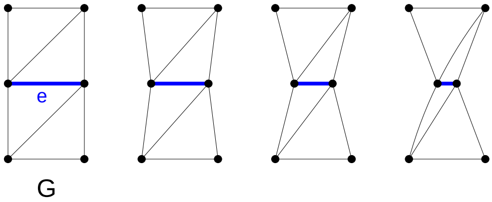

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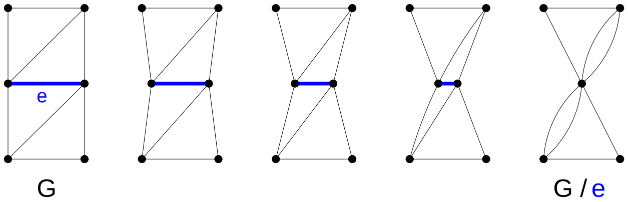

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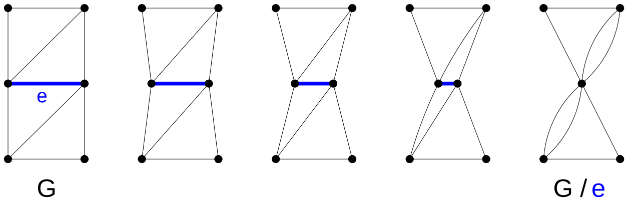

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- Therefore, we can calculate $\tau(G)$ recursively...
- ...but this is computationally inefficient (since it requires $2^{|E|}$ steps)...
- ...and, in general, is not useful for proving enumerative results like Cayley's formula.

The Matrix-Tree Theorem

$G = (V, E)$: connected graph without loops (parallel edges OK)
$V = \{1, 2, \dots, n\}$

Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

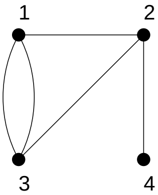
$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -(\#\text{ of edges between } i \text{ and } j) & \text{otherwise.} \end{cases}$$

- rank $L = n - 1$.

The Matrix-Tree Theorem

Example

$G =$



$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Matrix-Tree Theorem

The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} .$$

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$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} .$$

(2) Pick any $i \in \{1, \ldots, n\}$. Form the *reduced Laplacian* $\tilde{L}$ by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det \tilde{L} .$$

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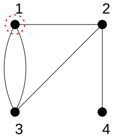
$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \tilde{L} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues: 0, 1, 4, 5

$$(1 \cdot 4 \cdot 5)/4 = 5$$

$$\det \tilde{L} = 5$$

The Chip-Firing Game

- Discrete dynamical system on graphs discovered independently by many: Biggs, Dhar, Merino, . . .

- Essentially equivalent to the *abelian sandpile model*, *dollar game*, . . .

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- Let $G = (V, E)$ be a simple graph, $V = \{0, 1, \ldots, n\}$. Each vertex i has a finite number c_i of poker chips.

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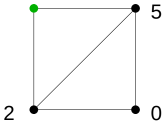
- Vertex 0, the *bank*, only fires if no other vertex can fire.
- Vertices other than the bank cannot go into debt
- State of the system $= \mathbf{c} = (c_1, \ldots, c_n)$
  (We don't care how many chips the bank has.)

The Chip-Firing Game

Bank



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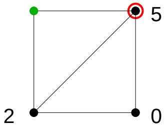

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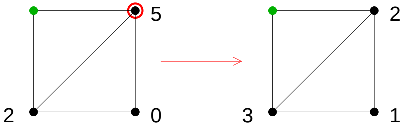

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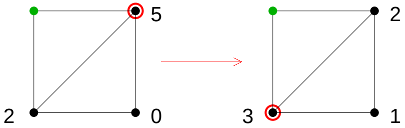

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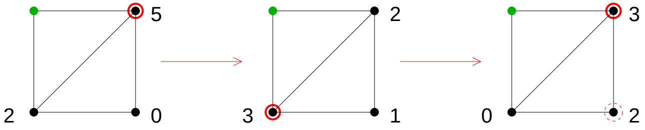

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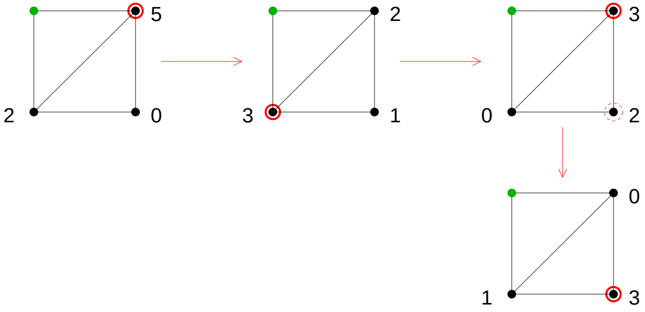

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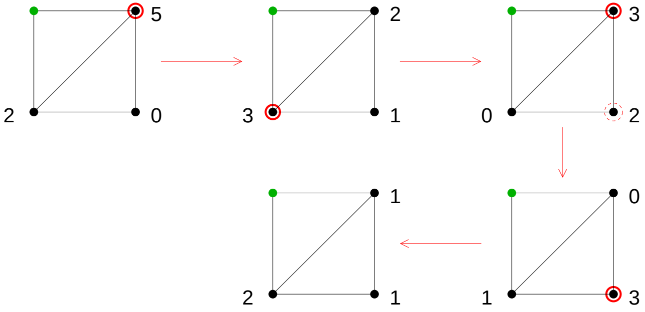

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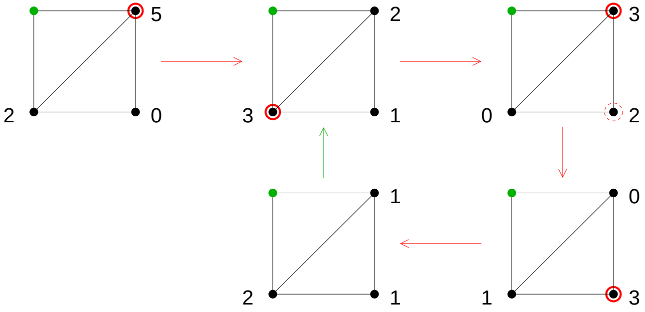

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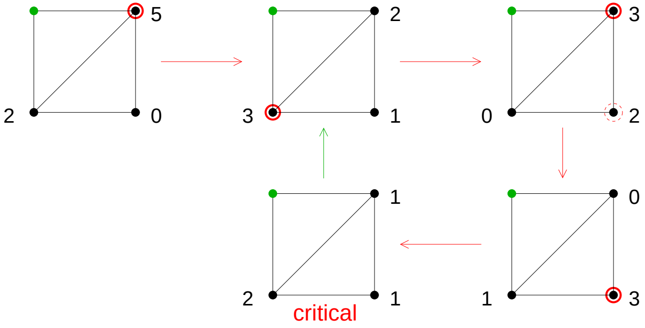

Chip-Firing and the Laplacian

- Recall: reduced Laplacian of G is $\tilde{L} = [\ell_{ij}]_{i,j=1\dots n}$, where

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Fact Each starting state $\mathbf{c}$ eventually leads to a unique critical state $Crit(\mathbf{c})$.

Chip-Firing and Trees

Call two state vectors $\mathbf{c}, \mathbf{c}'$ *firing-equivalent* if their difference is in the column space of $\tilde{L}$.

Fact $\mathbf{c}, \mathbf{c}'$ are firing-equivalent if and only if $Crit(\mathbf{c}) = Crit(\mathbf{c}')$.

Fact Number of critical states $= \det \tilde{L} = \tau(G)$.

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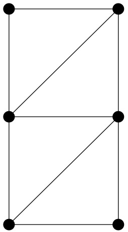

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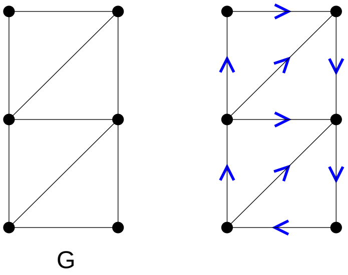

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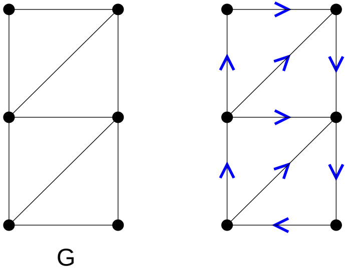

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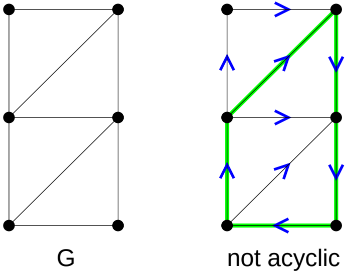

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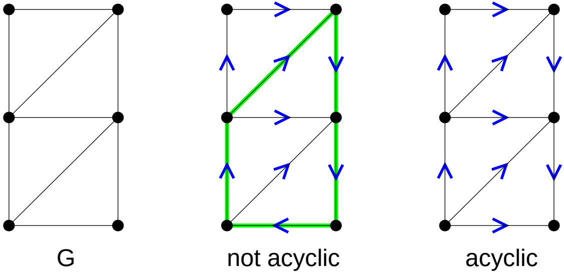

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Theorem $\alpha(G) = \alpha(G - e) + \alpha(G/e)$.

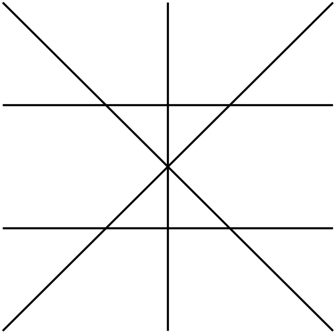
(Fact: Both $\alpha(G)$ and $\tau(G)$, as well as any other invariant satisfying a deletion-contraction recurrence, can be obtained from the *Tutte polynomial* $T_G(x, y)$.)

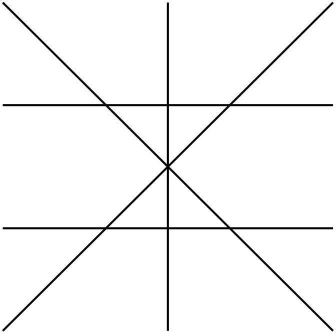
Hyperplane Arrangements

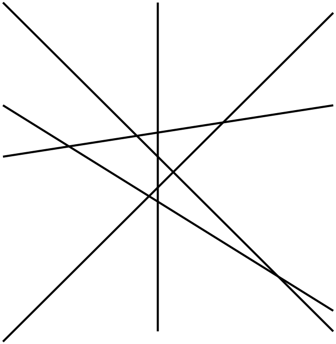
Definition A **hyperplane** H in $\mathbb{R}^n$ is an $(n-1)$-dimensional affine linear subspace.

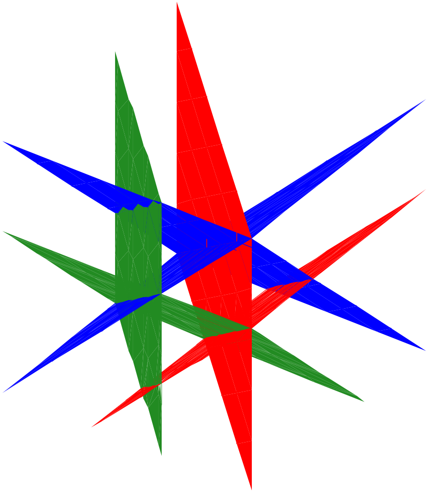
Definition A **hyperplane arrangement** $\mathcal{A} \subset \mathbb{R}^n$ is a finite collection of hyperplanes.

- $n = 1$: points on a line
- $n = 2$: lines on a plane
- $n = 3$: planes in 3-space









Counting Regions

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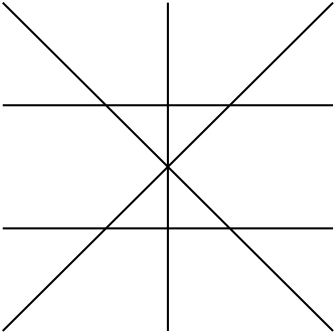

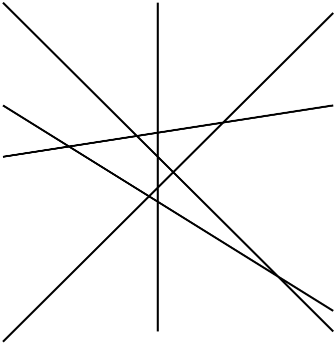

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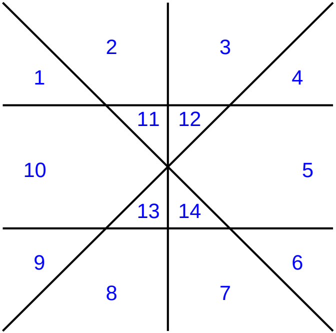
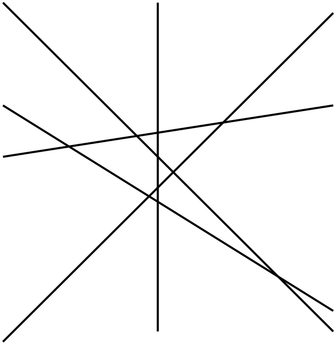


14 regions



16 regions

Counting Regions

Example $\mathcal{A} = n$ lines in $\mathbb{R}^2$

- $2n \;\leq\; r(\mathcal{A}) \;\leq\; 1 + \binom{n+1}{2}$

Example $\mathcal{A} = n$ coordinate hyperplanes in $\mathbb{R}^n$

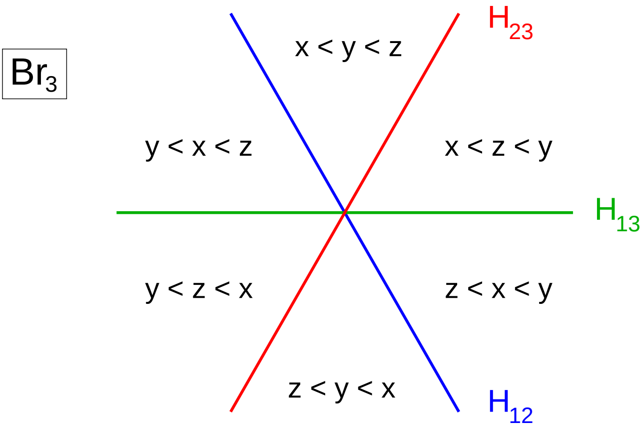
- Regions of $\mathcal{A}$ = orthants
- $r(\mathcal{A}) = 2^n$

The Braid Arrangement

The *braid arrangement* $Br_n \subset \mathbb{R}^n$ consists of the $\binom{n}{2}$ hyperplanes

$$\begin{aligned} H_{12} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\}, \\ H_{13} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\}, \\ &\dots \\ H_{n-1,n} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}. \end{aligned}$$

- $\mathbb{R}^n \setminus Br_n = \{\mathbf{x} \in \mathbb{R}^n \mid \text{all } x_i \text{ are distinct}\}$.
- **Problem:** Count the regions of Br_n.


Br3
H23
H13
H12
x < y < z
y < x < z
x < z < y
y < z < x
z < x < y
z < y < x

Graphic Arrangements

Let $G = (V, E)$ be a simple graph with $V = [n] = \{1, \ldots, n\}$. The *graphic arrangement* $\mathcal{A}_G \subset \mathbb{R}^n$ consists of the hyperplanes

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Theorem There is a bijection between regions of $\mathcal{A}_G$ and acyclic orientations of G. In particular,

$$r(\mathcal{A}_G) = \alpha(G).$$

(When $G = K_n$, the arrangement $\mathcal{A}_G$ is the braid arrangement.)

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The resulting orientation is acyclic.

Corollary $r(Br_n) = \alpha(K_n) = n!$.

Parking Functions

There are n parking spaces on a one-way street.

Cars $1, \ldots, n$ want to park in the spaces.

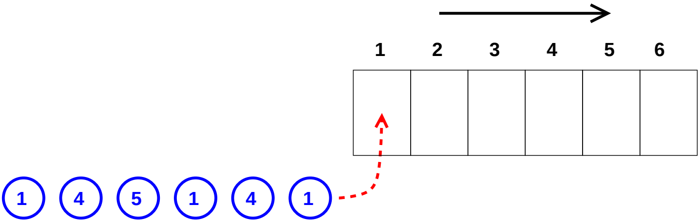
Each car has a preferred spot p_i.

Can all the cars park?

(Analogy: Hash table. . .)

Parking Functions

Example #1: $n = 6$; $(p_1, \ldots, p_6) = (1, 4, 1, 5, 4, 1)$



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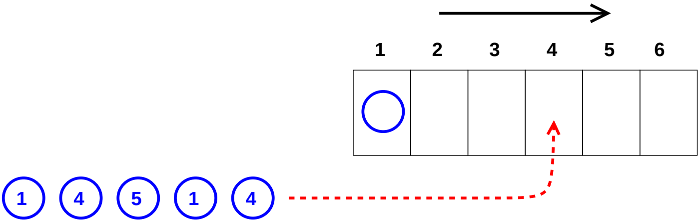

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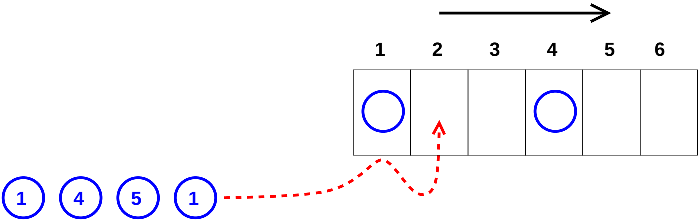

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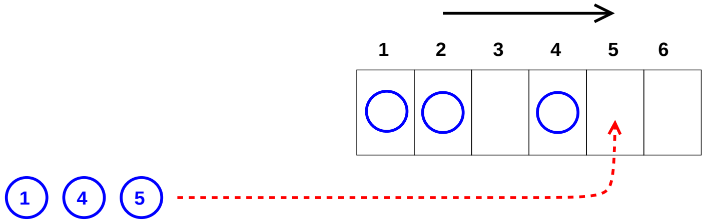

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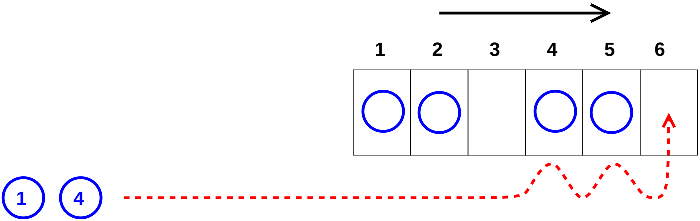

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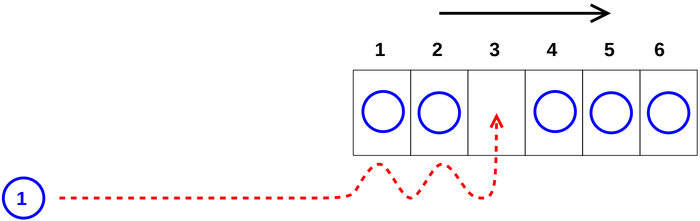

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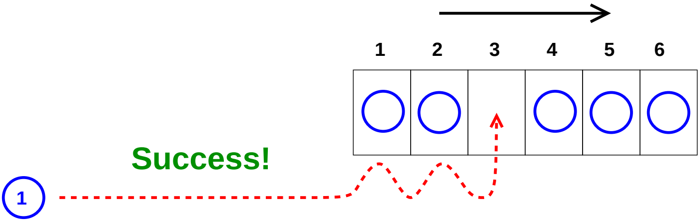

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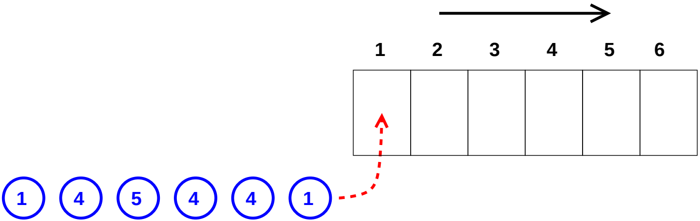

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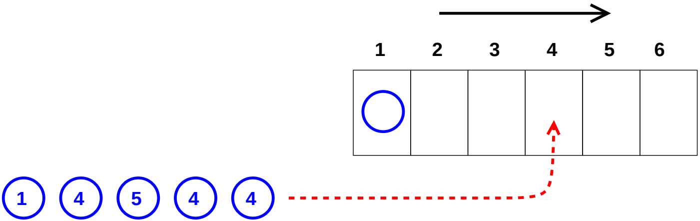

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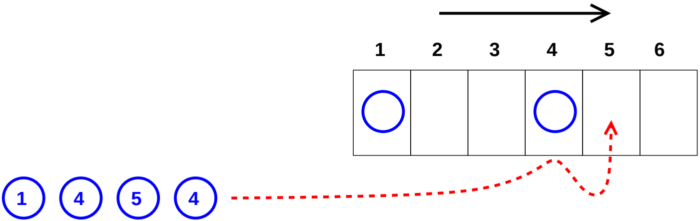

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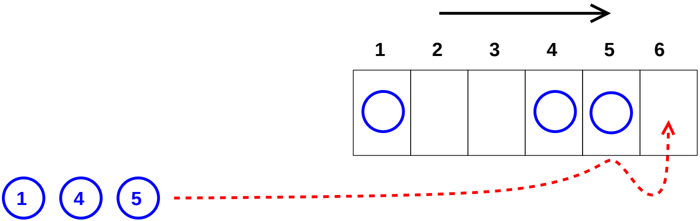

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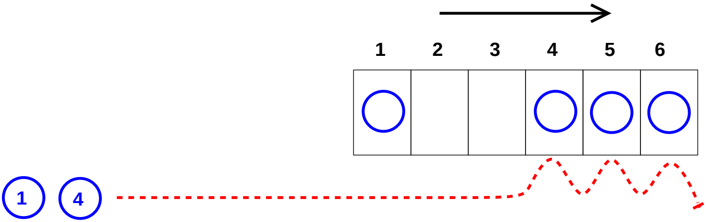

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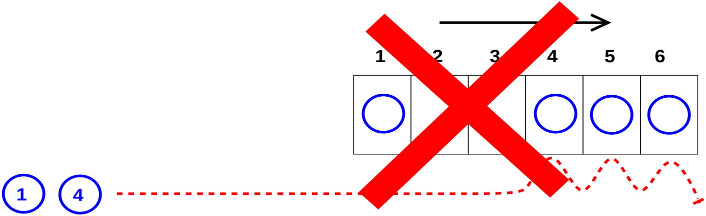

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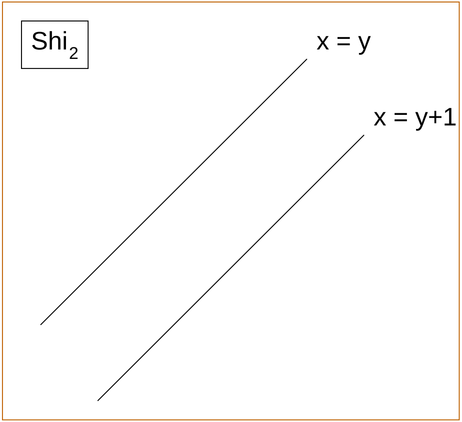
- In particular, parking functions are invariant up to permutation.
- The number of parking functions of length n is $(n+1)^{n-1}$.

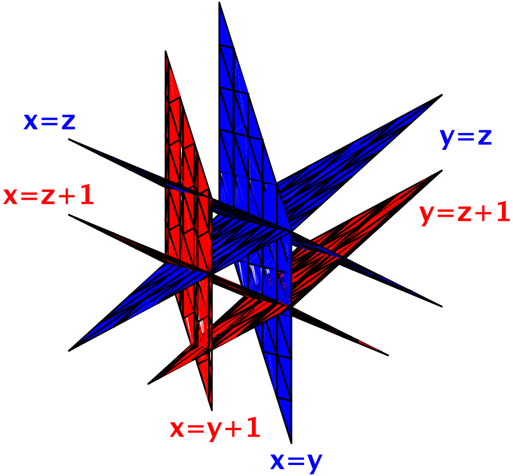
The Shi Arrangement

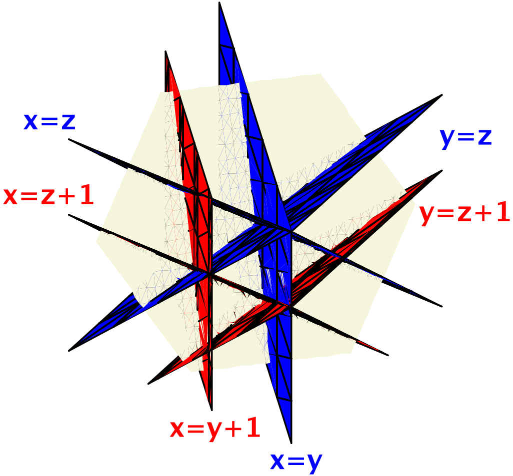
The *Shi arrangement* $Shi_n \subset \mathbb{R}^n$ consists of the $2\binom{n}{2}$ hyperplanes

$$\begin{array}{ll}
\{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2 + 1\}, \\
\{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3 + 1\}, \\
\dots & \\
\{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n + 1\}.
\end{array}$$

The Shi Arrangement




x=z
y=z
x=z+1
y=z+1
x=y+1
x=y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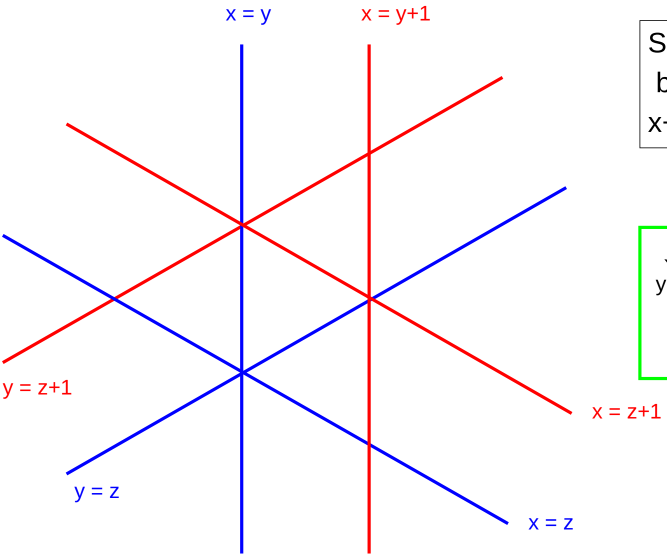
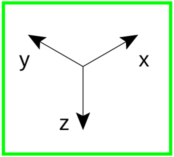


Slice of Shi_3
by plane
x+y+z = 0



The Shi Arrangement

Theorem The number of regions in Shi_n is $(n+1)^{n-1}$.

(Many proofs known: Shi, Athanasiadis-Linusson, Stanley . . .)

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Let $\mathbf{x} \in \mathbb{R}^n \setminus Shi_n$. For every $1 \leq i < j \leq n$:

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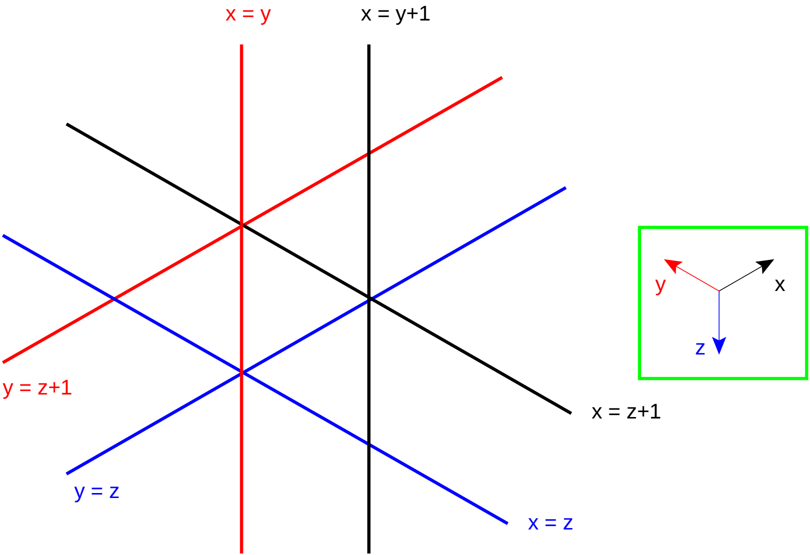
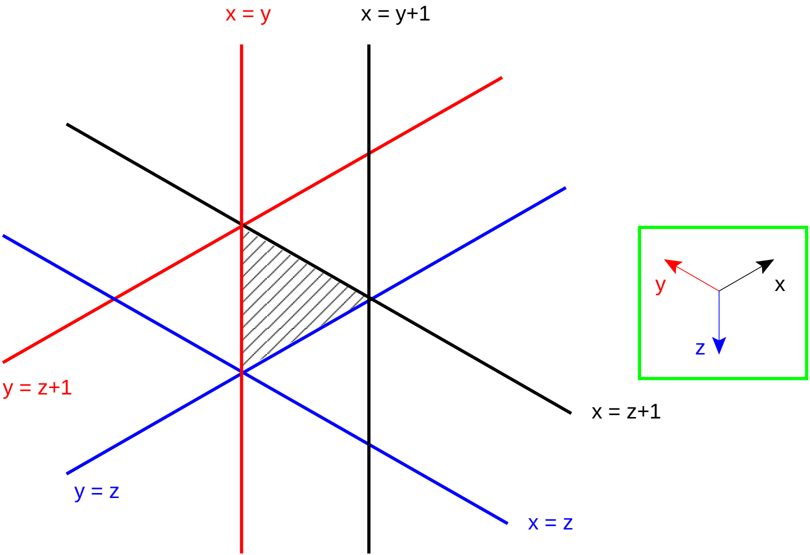
Example The score vector of $\mathbf{x} = (3.142,\ 2.010,\ 2.718)$ is $\mathbf{s} = (1, 0, 1)$.


x = y
x = y+1
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
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x = z
y
x
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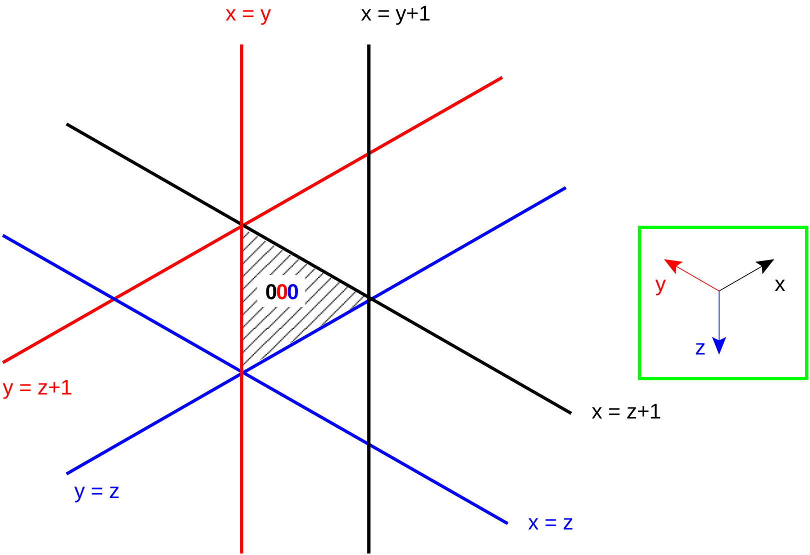
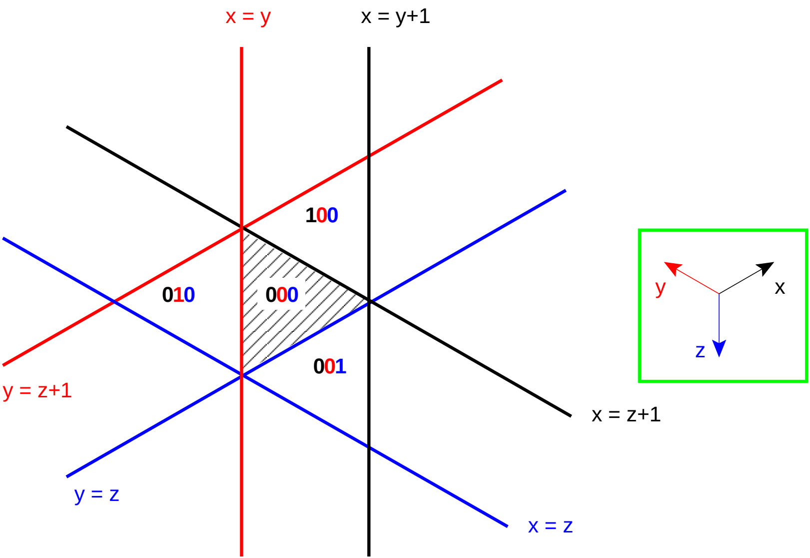
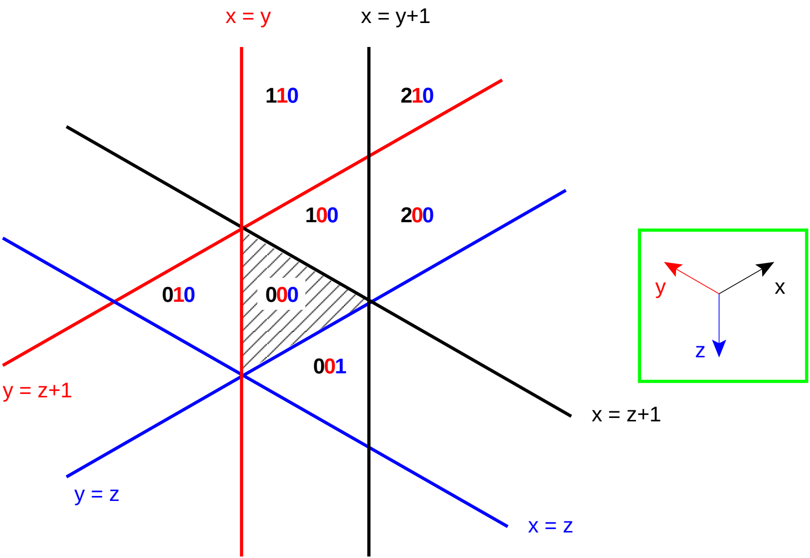
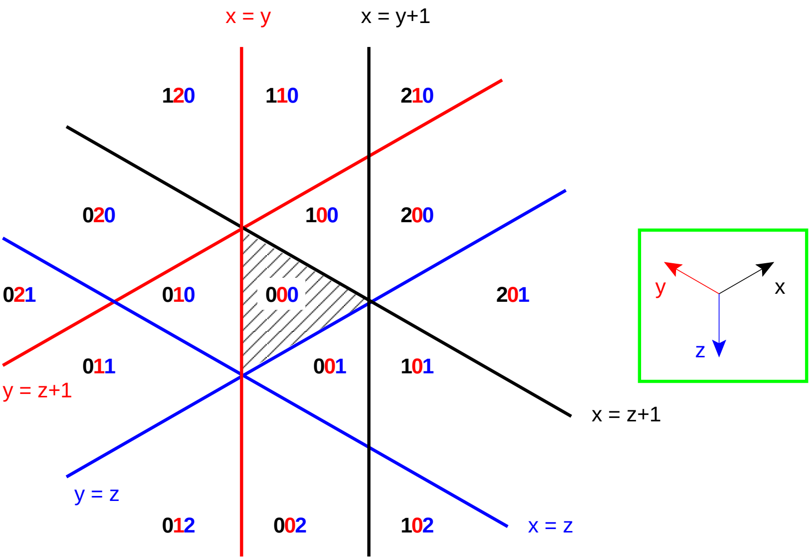

x = y
x = y+1
000
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
100
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
110
210
100
200
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
120
110
210
020
100
200
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010
000
201
011
001
101
y = z+1
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where d = distance, R_0 = base region.

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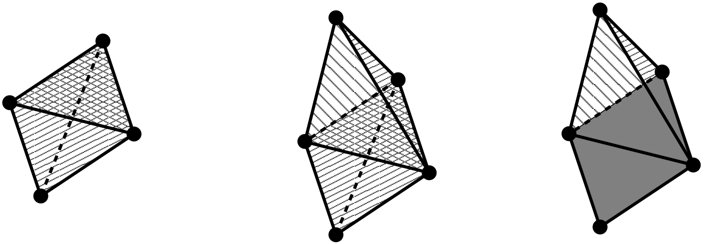
Example For $n=3$: $T_{K_4}(1,y) = 1+3y+6y^2+6y^3$.

Simplicial Complexes

Definition A *simplicial complex* is a space built out of

- vertices (dimension 0)
- edges (dimension 1)
- triangles (dimension 2)
- tetrahedra (dimension 3)
- higher-dimensional simplices

Simplicial complexes are the natural higher-dimensional analogues of graphs.



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- Hyperplane arrangements: ???