Informal Seminar on Stanley-Reisner Theory, UMN, Fall 2002
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Introduction and motivation for Stanley-Reisner rings, I
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1. Definitions

Definition 1. *Let V be a finite set of vertices. An abstract simplicial complex Δ on V is a subset of the power set 2^V which is closed under inclusion, that is,*

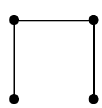
$$F \in \Delta,\ G \subset F \quad \Longrightarrow \quad G \in \Delta.$$

The elements of Δ are called *faces*. The *dimension* of a face, $\dim F$, is defined as $|F| - 1$.

We can often represent Δ pictorially. For instance, if $V = \{a, b, c, d\}$ and Δ is the abstract simplicial complex

$$\Delta = \{\emptyset, a, b, c, d, ab, bc, cd\} \tag{1}$$

(abbreviating the face $\{a\}$ by a, $\{a, b\}$ by ab, etc.), then the corresponding figure is



One can think of a, b, c, d as orthonormal basis vectors in $|V|$-space, so that the face ab (which has dimension 1) represents the affine span of the vectors a, b (which is a line segment), etc.

Fix a field $\mathbf{k}$, and let

$$S = \mathbf{k}[x_v \ :\ v \in V],$$

the (commutative) polynomial ring in variables corresponding to the vertices.

Definition 2. *The Stanley-Reisner ideal of Δ is*

$$I_\Delta := \left(\prod_{j=1}^{r} x_{v_{i_j}} \ :\ \{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta \right).$$

Note that I_Δ is a *monomial ideal* (that is, it is generated by monomials) and that the generators are *squarefree* (they are not divisible by the square of any variable). A minimal set of generators is given by the minimal nonfaces of Δ.

Definition 3. *The Stanley-Reisner ring (or face ring) of Δ is*

$$\mathbf{k}[\Delta] := S/I_\Delta.$$

Note that the set of monomials

$$\left\{ x_{v_1}^{e_1} \ldots x_{v_r}^{e_r} \ :\ \{v_1, \ldots, v_r\} \in \Delta,\ e_1, \ldots, e_r > 0 \right\}$$

is a basis for $\mathbf{k}[\Delta]$ as a $\mathbf{k}$-vector space. In particular, $\mathbf{k}[\Delta]$ is a graded ring.

For example, if Δ is the simplicial complex given in (1), then

$$I_\Delta = (ac, ad, bd)$$

(the minimal nonfaces of Δ) and $\mathbf{k}[\Delta]$ is the k-linear span of

$$\left\{ \begin{array}{ll} 1, a, a^2, a^3, \ldots, & ab, a^2b, ab^2, \ldots, \\ 1, b, b^2, b^3, \ldots, & bc, b^2c, bc^2, \ldots, \\ 1, c, c^2, c^3, \ldots, & cd, c^2d, cd^2, \ldots, \\ 1, d, d^2, d^3, \ldots & \end{array} \right\}. \tag{2}$$

This construction actually gives a bijection between simplicial complexes on V and ideals of S generated by squarefree monomials. The simplicial complex corresponding to such an ideal is its *Stanley-Reisner complex*.

2. Motivations

1. In algebraic geometry, one wants to study rings of the form $R = S/I$, where S is a polynomial ring over a field $\mathbf{k}$ and I is an ideal of S. That is, R is the coordinate ring of the affine algebraic variety defined by I. To study R using Stanley-Reisner rings, we may proceed as follows:

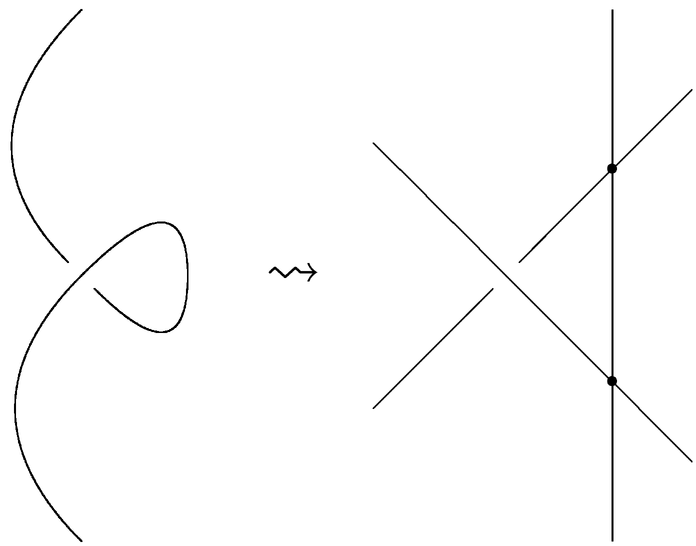
First, "deform" I as follows. Fix some monomial order $<$ on S and compute the *initial ideal* $\mathrm{in}_<(I) \subset S$ (this is equivalent to computing a Gröbner basis of I). By definition, $\mathrm{in}_<(I)$ is generated by monomials, However, the generators need not be squarefree, so a second step, *polarization*, may be required. The idea of this step is to get rid of high powers of variables by the following trick: if one of the generators of $\mathrm{in}_<(I)$ is, say, x^2, then we adjoin a new variable x', replace x^2 with xx', and mod out by $x - x'$. The result is a squarefree monomial ideal of some polynomial ring $S' \supset S$, which we may regard as the Stanley-Reisner ideal I_Δ of a simplicial complex Δ on the variables of S'. The ideal $\mathrm{in}_<(I)$ and its polarization are *very* closely related, so we don't have to worry too much about this second step.

The passage from I to $\mathrm{in}_<(I)$ does not preserve all structure, but it is pretty good (in the language of algebraic geometry, it is a *flat deformation*). Lots of geometric/ring invariants of R are closely related—often equal—to those of $\mathbf{k}[\Delta] = S'/I_\Delta$. For instance, the dimension, degree and Hilbert series of R are the same as for $\mathbf{k}[\Delta]$, and these can be computed combinatorially from Δ. In addition, some homological-type properties, such as Cohen-Macaulayness, can only get worse–e.g., if $\mathbf{k}[\Delta]$ is Cohen-Macaulay then so is R. (For an example, see the scribe's Ph.D. thesis.)

Here's an elementary example. Let $S = \mathbf{k}[a, b, c, d]$, $I = (ac - b^2, bd - c^2, ad - bc)$, and $R = S/I$. (In fact, $\mathrm{Proj}(R)$ is the twisted cubic, the image of the degree-3 Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ mapping $[s : t]$ to $[s^3 : s^2t : st^2 : t^3]$ in homogeneous coordinates.) The given generators of I form a Gröbner basis with respect to lexicographic order on the monomials of S, with $a < b < c < d$, so the initial ideal is

$$\mathrm{in}_<(I) = (ac, bd, ad),$$

the Stanley-Reisner ideal of the simplicial complex Δ given in (1). Geometrically, the twisted cubic is being "flattened out":



We compute the dimension, degree and Hilbert series of the twisted cubic using $\mathbf{k}[\Delta]$.

Fact: $\dim \mathbf{k}[\Delta] = 1 + \dim \Delta = 1 + \max\{\dim F \ : \ F \in \Delta\}$.

In this case, the largest faces have dimension 1, so $\dim R = 2$. It seems as though R should have dimension 1, but it is really the affine coordinate ring of the cone over the twisted cubic, so $\dim R = 2$ makes sense.

Fact: $\deg \mathbf{k}[\Delta] =$ number of facets (maximal faces) of Δ.

Here, that number is 3.

Now for the Hilbert series. By definition, this is

$$\mathrm{Hilb}(\mathbf{k}[\Delta], t) := \sum_{m \geq 0} (\dim_{\mathbf{k}} (\mathbf{k}[\Delta])_m)\, t^m$$

where $(\mathbf{k}[\Delta])_m$ denotes the mth graded piece of $\mathbf{k}[\Delta]$ (that is, the $\mathbf{k}$-linear span of the monomials of degree m. We have already written down a monomial basis (2) for $\mathbf{k}[\Delta]$. The minimal elements of the basis correspond to faces of Δ, and we have

$$\begin{aligned} \mathrm{Hilb}(\mathbf{k}[\Delta], t) &= 1 + 4\left(\frac{t}{1-t}\right) + 3\left(\frac{t^2}{(1-t)^2}\right) \\ &= \frac{1+2t}{(1-t)^2}. \end{aligned}$$

The dimension and degree can be extracted from the Hilbert series: the dimension is the order of the pole at $t = 1$ (here 2), and the degree is the sum of the coefficients in the numerator (here 3). The Hilbert series has the following combinatorial interpretation in terms of Δ.

Definition 4. *The f-vector of Δ is*

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{\dim \Delta}),$$

where f_i is the number of i-dimensional faces. (So $f_{-1} = 1$, since $\emptyset$ is the unique face of dimension -1, and f_0 is the number of vertices.)

Definition 5. *The h-vector of Δ is*

$$h(\Delta) = (h_0, h_1, \ldots, h_{\dim \Delta + 1}),$$

where h_i is defined as follows. You can use the formula

$$\sum_i h_i t^i = \sum_{i=0}^{\dim \Delta+1} f_{i-1} t^i (1-t)^{\dim \Delta - i+1}$$

(Bruns and Herzog, Lemma 5.1.8).

A more fun way to compute the h-vector is as follows. First, draw a triangle with the coefficients of the f-vector down the rightmost diagonal, and 1's down the leftmost diagonal:

$$\begin{array}{ccccccc}
 & & & f_{-1} & & & \\
 & & 1 & & f_0 & & \\
 & 1 & & \blacksquare & & f_1 & \\
1 & & \blacksquare & & \blacksquare & & f_2 \\
 & & & \vdots & & &
\end{array}$$

Then, fill in the $\blacksquare$'s down to the row below $f_{\dim \Delta}$ as though you were constructing Pascal's triangle—but subtracting instead of adding. That is, replace each $\blacksquare$ with the number to the northeast minus the number to the northwest. The bottom row will be the h-vector.

For example, for the complex Δ we have been working with, we start with

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 4 & & \\
 & 1 & & \blacksquare & & 3 & \\
\hline
1 & & \blacksquare & & \blacksquare & &
\end{array}$$

Filling in the boxes, we get

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 4 & & \\
 & 1 & & 3 & & 3 & \\
\hline
1 & & 2 & & 0 & &
\end{array}$$

The numbers below the line form the h-vector, in this case $(1, 2, 0)$. The trailing 0's are frequently dropped, so we would write $h(\Delta) = (1, 2)$.

Now back to the Hilbert series. The connection is the following:

$$\begin{aligned}
\operatorname{Hilb}(\mathbf{k}[\Delta], t) &:= \sum_{m \geq 0} (\dim_{\mathbf{k}}(\mathbf{k}[\Delta])_m)\, t^m \\
&= \sum_{i \geq 0} f_i(\Delta) \left(\frac{t}{1-t} \right)^{i+1} \\
&= \frac{\sum_{i=0}^{\dim \Delta+1} h_i t^i}{(1-t)^{\dim \Delta+1}}.
\end{aligned}$$