Informal Seminar on Stanley-Reisner Theory, UMN, Fall 2002
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Shellable complexes are Cohen-Macaulay
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I'm going to prove the following (Theorem 5.1.13 in Bruns and Herzog):

Theorem: Let Δ be a shellable simplicial complex. Then the Stanley-Reisner ring $\mathbf{k}[\Delta]$ is Cohen-Macaulay (for all fields $\mathbf{k}$).

We need the following preliminary results. Let R be a $\mathbb{N}^n$-graded ring and I, J graded ideals of R. (We may as well take $R = \mathbf{k}[x_1, \ldots, x_n]$ and I, J monomial ideals.)

Lemma 1: We have an exact sequence

$$0 \;\to\; R/(I \cap J) \;\xrightarrow{\alpha}\; R/I \oplus R/J \;\xrightarrow{\beta}\; R/(I+J) \;\to\; 0.$$

where

$$\begin{aligned} \alpha(a + (I+J)) &= (a+I, a-J), \\ \beta(a+I, b+J) &= (a+b) + (I+J). \end{aligned}$$

Proof: α obviously has zero kernel, and β is surjective by the Chinese Remainder Theorem. Finally,

$$\begin{aligned} \operatorname{im}\alpha &= I/(I \cap J) \;\oplus\; J/(I \cap J) \qquad \text{and} \\ \ker\beta &= (I+J)/I \;\oplus\; (I+J)/J \end{aligned}$$

which are equal by basic group theory. □

Lemma 2: If R/I and R/J are Cohen-Macaulay of dimension d and $R/(I+J)$ is Cohen-Macaulay of dimension $d-1$, then $R/(I \cap J)$ is Cohen-Macaulay of dimension d.

Proof: It is easy to check that $R/I \oplus R/J$ is Cohen-Macaulay of dimension d. Recall that $H^i(M)$ is nonzero iff depth $M \leq i \leq \dim M$ (with equality if M is Cohen-Macaulay). So we want to show that

(1) $$H^i(R/(I \cap J) \neq 0 \quad\iff\quad i = d.$$

Apply the long exact sequence of local cohomology to the short exact sequence of Lemma 1, obtaining

(2)
$$\ldots \to H^{i-1}(R/I \oplus R/J) \to H^{i-1}(R/(I+J)) \to H^i(R/(I \cap J) \to H^i(R/I \oplus R/J) \to H^i(R/(I+J)) \to \ldots$$

If $i \neq d$, then the second and fourth terms in (2) are zero, so $H^i(R/(I \cap J) = 0$. If $i = d$, then the first and fifth terms are zero and the second and fourth terms are nonzero, so $H^i(R/(I \cap J) \neq 0$, giving (1). □

Proof of the theorem: Let Δ be shellable of dimension $d-1$ on vertices $\{v_1, \ldots, v_n\}$, with $F_1, \ldots, F_m$ a shelling order on the facets. That is, for all $1 \leq j \leq m+1$, the complex $\langle F_{j+1} \rangle \cap \Delta_j$ is generated by some nonempty set of maximal proper faces of F_{j+1}, where $\Delta_j = \langle F_1, \ldots, F_j \rangle$. For each facet F_j, define an ideal

$$P_j = (x_i \;:\; v_i \notin F_j);$$

it is easy to verify that P_j is prime and that

$$I_\Delta = \bigcap_{j=1}^{m} P_j.$$

We will show by induction that for every j, the ring $\mathbf{k}[\Delta_j]$ is Cohen-Macaulay. If $j = 1$ then $\Delta_j = \langle F_1 \rangle$ is a simplex, so $\mathbf{k}[\Delta_j]$ is the polynomial ring on the variables $\{x_i : v_i \in F_1\}$.

For $j > 1$, let $I = I_{\Delta_{j-1}}$ and $J = P_j$. Then by Lemma 1 we have a short exact sequence

(3) $$0 \;\to\; \mathbf{k}[\Delta_j] \;\to\; \mathbf{k}[\Delta_{j-1}] \oplus \mathbf{k}[\langle F_j \rangle] \;\to\; \mathbf{k}[\langle F_j \rangle \cap \Delta_{j-1}] \;\to\; 0.$$

By the definition of shellability, $\langle F_j \rangle \cap \Delta_{j-1} = \langle G_1, \ldots, G_l \rangle$, where $G_k = F_j \setminus \{x_{h_k}\}$ for $1 \leq k \leq l$. Therefore

$$\mathbf{k}[\langle F_j \rangle \cap \Delta_{j-1}] \;=\; \mathbf{k}[x_i \;:\; v_i \in F_j] \;/\; \prod_{k=1}^{l} x_{h_k},$$

which is Cohen-Macaulay of dimension $d-1$. Moreover, $\mathbf{k}[\Delta_{j-1}]$ and $\mathbf{k}[\langle F_j \rangle]$ are Cohen-Macaulay of dimension d (the former by induction, the latter because it is a polynomial ring). So $\mathbf{k}[\Delta_j]$ is Cohen-Macaulay by Lemma 2. $\square$