SIMPLICIAL MATRIX-TREE THEOREMS

ART M. DUVAL, CAROLINE J. KLIVANS, AND JEREMY L. MARTIN

Abstract. We generalize the definition and enumeration of spanning trees from the setting of graphs to that of arbitrary-dimensional simplicial complexes Δ, extending an idea due to G. Kalai. We prove a simplicial version of the Matrix-Tree Theorem that counts simplicial spanning trees, weighted by the squares of the orders of their top-dimensional integral homology groups, in terms of the Laplacian matrix of Δ. As in the graphic case, one can obtain a more finely weighted generating function for simplicial spanning trees by assigning an indeterminate to each vertex of Δ and replacing the entries of the Laplacian with Laurent monomials. When Δ is a shifted complex, we give a combinatorial interpretation of the eigenvalues of its weighted Laplacian and prove that they determine its set of faces uniquely, generalizing known results about threshold graphs and unweighted Laplacian eigenvalues of shifted complexes.

1. Introduction

This article is about generalizing the Matrix-Tree Theorem from graphs to simplicial complexes.

1.1. **The classical Matrix-Tree Theorem.** We begin by reviewing the classical case; for a more detailed treatment, see, e.g., [8]. Let G be a finite, simple, undirected graph with vertices $V(G) = [n] = \{1, 2, \ldots, n\}$ and edges $E(G)$. A *spanning subgraph* of G is a graph T with $V(T) = V(G)$ and $E(T) \subseteq E(G)$; thus a spanning subgraph may be specified by its edge set. A spanning subgraph T is a *spanning tree* if (a) T is acyclic; (b) T is connected; and (c) $|E(T)| = |V(T)| - 1$. It is a fundamental property of spanning trees (the "two-out-of-three theorem") that any two of these three conditions together imply the third.

The *Laplacian* of G is the $n \times n$ symmetric matrix $L = L(G)$ with entries

$$L_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -1 & \text{if } i, j \text{ are adjacent}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\deg_G(i)$ is the degree of vertex i (the number of edges having i as an endpoint). Equivalently, $L = \partial\partial^*$, where ∂ is the (signed) vertex-edge incidence matrix and ∂^* is its transpose. If we regard G as a one-dimensional simplicial complex, then ∂ is just the simplicial boundary map from 1-faces to 0-faces, and ∂^* is the simplicial coboundary map. The matrix L is symmetric, hence diagonalizable, so it has n real eigenvalues (counting multiplicities). The number of nonzero eigenvalues of L is $n - c$, where c is the number of components of G.

The Matrix-Tree Theorem, first observed by Kirchhoff [22] in his work on electrical circuits (modern references include [8], [29] and [34, Chapter 5]), expresses the number $\tau(G)$ of spanning trees of G in terms of L. The theorem has two equivalent formulations.

Theorem 1.1 (**Classical Matrix-Tree Theorem**). *Let G be a connected graph with n vertices, and let L be its Laplacian matrix.*

Date: August 14, 2008.

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05E99, 05C05, 05C50, 15A18, 57M15.

Key words and phrases. Simplicial complex, spanning tree, tree enumeration, Laplacian, spectra, eigenvalues, shifted complex.

Second author partially supported by NSF VIGRE grant DMS-0502215. Third author partially supported by an NSA Young Investigators Grant.

(1) *If the eigenvalues of L are $\lambda_0 = 0, \lambda_1, \ldots, \lambda_{n-1}$, then*

$$\tau(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

(2) *For $1 \leq i \leq n$, let L_i be the* reduced Laplacian *obtained from L by deleting the i^{th} row and i^{th} column. Then*

$$\tau(G) = \det L_i.$$

Well-known corollaries of the Matrix-Tree Theorem include Cayley's formula [9]

$$\tau(K_n) = n^{n-2} \tag{1}$$

where K_n is the complete graph on n vertices, and Fiedler and Sedláçek's formula [16]

$$\tau(K_{n,m}) = n^{m-1}m^{n-1} \tag{2}$$

where $K_{n,m}$ is the complete bipartite graph on vertex sets of sizes n and m.

The Matrix-Tree Theorem can be refined by introducing an indeterminate $e_{ij} = e_{ji}$ for each pair of vertices i, j, setting $e_{ij} = 0$ if i, j do not share a common edge. The *weighted Laplacian* $\hat{L}$ is then defined as the $n \times n$ matrix with entries

$$\hat{L}_{ij} = \begin{cases} \sum_{k=1}^{n} e_{ik} & \text{if } i = j, \\ -e_{ij} & \text{if } i, j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (**Weighted Matrix-Tree Theorem**). *Let G be a graph with n vertices, and let $\hat{L}$ be its weighted Laplacian matrix.*

(1) *If the eigenvalues of $\hat{L}$ are $\hat{\lambda}_0 = 0, \hat{\lambda}_1, \ldots, \hat{\lambda}_{n-1}$, then*

$$\sum_{T \in \mathcal{T}(G)} \prod_{ij \in T} e_{ij} = \frac{\hat{\lambda}_1 \cdots \hat{\lambda}_{n-1}}{n}$$

where $\mathcal{T}(G)$ is the set of all spanning trees of G.

(2) *For $1 \leq i \leq n$, let $\hat{L}_i$ be the* reduced weighted Laplacian *obtained from $\hat{L}$ by deleting the i^{th} row and i^{th} column. Then*

$$\sum_{T \in \mathcal{T}(G)} \prod_{ij \in T} e_{ij} = \det \hat{L}_i.$$

By making appropriate substitutions for the indeterminates e_{ij}, it is often possible to obtain finer enumerative information than merely the number of spanning trees. For instance, when $G = K_n$, introducing indeterminates $x_1, \ldots, x_n$ and setting $e_{ij} = x_i x_j$ for all i, j yields the *Cayley-Prüfer Theorem*, which enumerates spanning trees of K_n by their degree sequences:

$$\sum_{T \in \mathcal{T}(G)} \prod_{i=1}^{n} x_i^{\deg_T(i)} = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}. \tag{3}$$

Note that Cayley's formula (1) can be recovered from the Cayley-Prüfer Theorem by setting $x_1 = \cdots = x_n = 1$.

1.2. **Simplicial spanning trees and how to count them.** To extend the scope of the Matrix-Tree Theorem from graphs to simplicial complexes, we must first say what "spanning tree" means in arbitrary dimension. Kalai [20] proposed a definition that replaces the acyclicity, connectedness,

and edge-count conditions with their analogues in simplicial homology. Our definition adapts Kalai's definition to a more general class of simplicial complexes.[1]

Let Δ be a d-dimensional simplicial complex, and let $\Upsilon \subset \Delta$ be a subcomplex containing all faces of Δ of dimension $< d$. We say that Υ is a *simplicial spanning tree* of Δ if the following three conditions hold:

$$\tilde{H}_d(\Upsilon, \mathbb{Z}) = 0, \tag{4a}$$

$$|\tilde{H}_{d-1}(\Upsilon, \mathbb{Z})| < \infty, \text{ and} \tag{4b}$$

$$f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta), \tag{4c}$$

where $\tilde{H}_i$ denotes reduced simplicial homology (for which see, e.g., [18, §2.1]). (The conditions (4a) and (4b) were introduced by Kalai in [20], while (4c) is more general, as we will explain shortly.) When $d = 1$, the conditions (4a)…(4c) say respectively that Υ is acyclic, connected, and has one fewer edge than it has vertices, recovering the definition of the spanning tree of a graph. Moreover, as we will show in Proposition 3.5, any two of the three conditions together imply the third.

A graph G has a spanning tree if and only if G is connected. The corresponding condition for a simplicial complex Δ of dimension d is that $\tilde{H}_i(\Delta, \mathbb{Q}) = 0$ for all $i < d$; that is, Δ has the rational homology type of a wedge of d-dimensional spheres. We will call such a complex *acyclic in positive codimension*, or APC for short. This condition, which we will assume throughout the rest of the introduction, is much weaker than Cohen-Macaulayness (by Reisner's theorem [30]), and therefore encompasses many complexes of combinatorial interest, including all connected graphs, simplicial spheres, shifted, matroid, and Ferrers complexes, and some chessboard and matching complexes.

For $k \le d$, let $\partial = \partial_k$ be the kth simplicial boundary matrix of Δ (with rows and columns indexed respectively by $(k-1)$-dimensional and k-dimensional faces of Δ), and let ∂^* be its transpose. The *(kth up-down) Laplacian* of Δ is $L = \partial\partial^*$; this can be regarded either as a square matrix of size $f_{k-1}(\Delta)$ or as a linear endomorphism on $(k-1)$-chains of Δ. Define invariants

$$\pi_k = \pi_k(\Delta) = \text{ product of all nonzero eigenvalues of } L,$$

$$\tau_k = \tau_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon)|^2,$$

where $\mathcal{T}_k(\Delta)$ denotes the set of all k-trees of Δ (that is, simplicial spanning trees of the k-skeleton of Δ).

Kalai [20] studied these invariants in the case that Δ is a simplex on n vertices, and proved the formula

$$\tau_k(\Delta) = n^{\binom{n-2}{k}} \tag{5}$$

(of which Cayley's formula (1) is the special case $k = 1$). Kalai also proved a natural weighted analogue of (5) enumerating simplicial spanning trees by their degree sequences, thus generalizing the Cayley-Prüfer Theorem (3).

Given disjoint vertex sets $V_1, \ldots, V_r$ ("color classes"), the faces of the corresponding *complete colorful complex* Γ are those sets of vertices with no more than one vertex of each color. Equivalently, Γ is the simplicial join $V_1 * V_2 * \cdots * V_r$ of the 0-dimensional complexes V_i. Adin [1] extended Kalai's work by proving a combinatorial formula for $\tau_k(\Gamma)$, which we shall not reproduce here, for every $1 \le k < r$. Note that when $r = 2$, the complex Γ is a complete bipartite graph, and if $|V_i| = 1$ for all i, then Γ is a simplex. Thus both (2) and (5) can be recovered from Adin's formula.

Kalai's and Adin's beautiful formulas inspired us to look for more results about simplicial spanning tree enumeration, and in particular to formulate a simplicial version of the Matrix-Tree Theorem

[1]There are many other definitions of "simplicial tree" in the literature, depending on which properties of trees one wishes to extend; see, e.g., [4, 10, 15, 19, 28]. By adopting Kalai's idea, we choose a definition that lends itself well to enumeration. The closest to ours in spirit is perhaps that of Masbaum and Vaintrob [28], whose main result is a Matrix-Tree-like theorem enumerating a different kind of 2-dimensional tree using Pfaffians rather than Laplacians.

that could be applied to as broad a class of complexes as possible. Our first main result generalizes the Matrix-Tree Theorem to all APC simplicial complexes.

Theorem 1.3 (**Simplicial Matrix-Tree Theorem**). *Let Δ be a d-dimensional APC simplicial complex. Then:*

(1) *We have*
$$\pi_d(\Delta) = \frac{\tau_d(\Delta)\tau_{d-1}(\Delta)}{|\tilde{H}_{d-2}(\Delta)|^2}.$$

(2) *Let U be the set of facets of a $(d-1)$-SST of Δ, and let L_U be the reduced Laplacian obtained by deleting the rows and columns of L corresponding to U. Then*
$$\tau_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Delta_U)|^2} \det L_U.$$

We will prove these formulas in Section 4.

In the special case $d = 1$, the number $\tau_1(\Delta)$ is just the number of spanning trees of the graph Δ, recovering the classical Matrix-Tree Theorem. When $d \geq 2$, there can exist spanning trees with finite but nontrivial homology groups (the simplest example is the real projective plane). In this case, $\tau_k(\Delta)$ is greater than the number of spanning trees, because these "torsion trees" contribute more than 1 to the count. This phenomenon was first observed by Bolker [7], and arises also in the study of cyclotomic matroids [26] and cyclotomic polytopes [3].

The Weighted Matrix-Tree Theorem also has a simplicial analogue. Introduce an indeterminate x_F for each facet (maximal face) $F \in \Delta$, and for every set T of facets define monomials $x_T = \prod_{F \in T} x_F$ and $X_T = x_T^2$. Construct the *weighted boundary matrix* $\hat{\partial}$ by multiplying each column of ∂ by x_F, where F is the facet of Δ corresponding to that column. Let $\hat{\pi}_k$ be the product of the nonzero eigenvalues of $\hat{L}^{\mathrm{ud}}_{\Delta,k-1}$, and let
$$\hat{\tau}_k = \hat{\tau}_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon)|^2 X_\Upsilon.$$

Theorem 1.4 (**Weighted Simplicial Matrix-Tree Theorem**). *Let Δ be a d-dimensional APC simplicial complex. Then:*

(1) *We have*[2]
$$\hat{\pi}_d(\Delta) = \frac{\hat{\tau}_d(\Delta)\tau_{d-1}(\Delta)}{|\tilde{H}_{d-2}(\Delta)|^2}.$$

(2) *Let U be the set of facets of a $(d-1)$-SST of Δ, and let $\hat{L}_U$ be the reduced Laplacian obtained by deleting the rows and columns of $\hat{L}$ corresponding to U. Then*
$$\hat{\tau}_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Delta_U)|^2} \det \hat{L}_U.$$

We will prove these formulas in Section 5.

Setting $x_F = 1$ for all F in Theorem 1.4 recovers Theorem 1.3. In fact, more is true; setting $x_F = 1$ in the multiset of eigenvalues of the weighted Laplacian (reduced or unreduced) yields the eigenvalues of the corresponding unweighted Laplacian. If the complex Δ is *Laplacian integral*, that is, its Laplacian matrix has integer eigenvalues, then we can hope to find a combinatorial interpretation of the factorization of $\hat{\tau}_d(\Delta)$ furnished by Theorem 1.4. An important class of Laplacian integral simplicial complexes are the *shifted complexes*.

[2]Despite appearances, there are no missing hats on the right-hand side of this formula! Only $\tau_d(\Delta)$ has been replaced with its weighted analogue; $\tau_{d-1}(\Delta)$ is still just an integer.

1.3. **Results on shifted complexes.** Let $p \le q$ be integers, and let $[p,q] = \{i \in \mathbb{Z} \colon p \le i \le q\}$. A simplicial complex Σ on vertex set $[p,q]$ is *shifted* if the following condition holds: whenever $i < j$ are vertices and $F \in \Sigma$ is a face such that $i \notin F$ and $j \in F$, then $F \backslash \{j\} \cup \{i\} \in \Sigma$. Equivalently, define the *componentwise partial order* $\preceq$ on finite sets of positive integers as follows: $A \preceq B$ whenever $A = \{a_1 < \cdots < a_m\}$, $B = \{b_1 < \cdots < b_m\}$, and $a_i \le b_i$ for all i. Then a complex is shifted precisely when it is an order ideal with respect to the componentwise partial order. (See [33, chapter 3] for general background on partially ordered sets.)

Shifted complexes were used by Björner and Kalai [5] to characterize the f-vectors and Betti numbers of all simplicial complexes. Shifted complexes are also one of a small handful of classes of simplicial complexes whose Laplacian eigenvalues are known to be integral. In particular, Duval and Reiner [13, Thm. 1.1] proved that the Laplacian eigenvalues of a shifted complex Σ on $[p,q]$ are given by the conjugate of the partition $(d_p, d_{p+1}, \ldots, d_q)$, where d_i is the degree of vertex i, that is, the number of facets containing it.

In the second part of the article, Sections 6–10, we study factorizations of the weighted spanning tree enumerator of Σ under the *combinatorial fine weighting*

$$x_F = \prod_{i=1}^{k+1} x_{i,v_i}$$

(described in more detail in Section 6), where $F = \{v_1 < \cdots < v_{k+1}\}$ is a k-dimensional face of Σ. Thus the term of $\tau_k(\Sigma)$ corresponding to a particular simplicial spanning tree of Σ contains more precise information than its vertex degrees alone (which can be recovered by further setting $x_{i,j} = x_j$ for all i, j).

For integer sets A and B as above, we call the ordered pair (A, B) a *critical pair* of Σ if $A \in \Sigma$, $B \notin \Sigma$, and B covers A in the componentwise order. That is, $B = \{a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_m\}$ for some $i \in [m]$. The *long signature* of (A, B) is the ordered pair $\bar{\sigma}(A, B) = (S, T)$, where $S = \{a_1, \ldots, a_{i-1}\}$ and $T = [p, a_i]$. The corresponding *z-polynomial* is defined as

$$z(S,T) = \frac{1}{\uparrow X_S} \sum_{j \in T} X_{S \cup j}$$

where $X_S = x_S^2$ for each S, and the operator $\uparrow$ is defined by $\uparrow(x_{i,j}) = x_{i+1,j}$. (See Section 8.1 for more details, and Example 1.7 for an example.) The set of critical pairs is especially significant for a shifted family (and by extension, for a shifted complex). Since a shifted family is just an order ideal with respect to the componentwise partial order $\preceq$, the critical pairs identify the frontier between members and non-members of $\mathcal{F}$ in the Hasse diagram of $\preceq$. (See Example 1.7 or [23] for more details.)

Thanks to Theorem 1.4, the enumeration of SST's of a shifted complex reduces to computing the determinant of the reduced combinatorial finely-weighted Laplacian. We show in Section 6 how this computation reduces to the computation of the eigenvalues of the *algebraic finely weighted Laplacian*. This modification of the combinatorial fine weighting, designed to endow the chain groups of Σ with the structure of an algebraic chain complex, is described in detail in Section 6.3. Its eigenvalues turn out to be precisely the z-polynomials associated with critical pairs.

Theorem 1.5. *Let Σ be a d-dimensional shifted complex, and let $0 \le i \le d$. Then the eigenvalues of the algebraic finely weighted up-down Laplacian $\mathbf{L}^{\mathrm{ud}}_{\Delta,i}$ are precisely $\uparrow^{d-i}(z(S,T))$, where (S,T) ranges over all long signatures of critical pairs of i-dimensional faces of Σ.*

In turn, the z-polynomials are the factors of the weighted simplicial spanning tree enumerator $\hat{\tau}_d$.

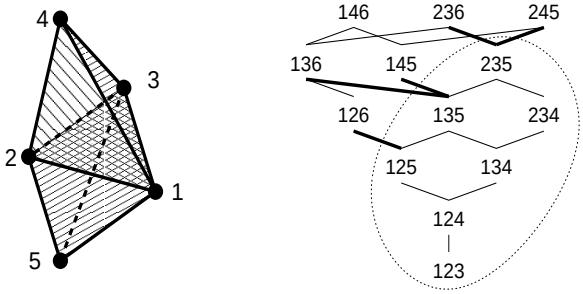
Theorem 1.6. *Let Σ be a d-dimensional shifted complex with initial vertex p. Then:*

$$\hat{\tau}_d(\Sigma) = \left(\prod_{F\in\Lambda_{d-1}} X_{\tilde{F}}\right)\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)} \frac{z(S,\tilde{T})}{X_{1,p}}\right)$$
$$= \left(\prod_{F\in\Lambda_{d-1}} X_{\tilde{F}}\right)\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)} \frac{\sum_{j\in\tilde{T}} X_{S\cup j}}{X_{\tilde{S}}}\right)$$

where $\tilde{F} = F \cup \{p\}$; $\Delta = \operatorname{del}_p \Sigma = \{F\backslash\{p\} \colon F \in \Sigma\}$; and $\Lambda = \operatorname{link}_p \Sigma = \{F \colon p \notin F,\ \tilde{F} \in \Sigma\}$.

Theorems 1.5 and 1.6 are proved in Sections 8 and 9, respectively.

Example 1.7. As an example to which we will return repeatedly, consider the *equatorial bipyramid*, the two-dimensional shifted complex B with vertices [5] and facets 123, 124, 125, 134, 135, 234, 235. A geometric realization of B is shown in the figure on the left below. The figure on the right illustrates how the facets of B can be regarded as an order ideal. The boldface lines indicate critical pairs.



The Laplacian eigenvalues corresponding to the critical pairs of B are as follows:

Critical pair	$(125, 126)$	$(135, 136)$	$(135, 145)$	$(235, 236)$	$(235, 245)$
Eigenvalue	$z(12, 12345)$	$z(13, 12345)$	$z(1, 123)$	$z(23, 12345)$	$z(2, 123)$

To show one of these eigenvalues in more detail,

$$z(13, 12345) = \frac{X_{1,1}X_{2,1}X_{3,3} + X_{1,1}X_{2,2}X_{3,3} + X_{1,1}X_{2,3}X_{3,3} + X_{1,1}X_{2,3}X_{3,4} + X_{1,1}X_{2,3}X_{3,5}}{X_{2,1}X_{3,3}}.$$

The eigenvalues of this complex are explained in more detail in Section 8.4. Its spanning trees are enumerated in Examples 9.1 (fine weighting) and 9.3 (coarse weighting).

We prove Theorem 1.5 by exploiting the recursive structure of shifted complexes. As in [13], we begin by calculating the algebraic finely weighted eigenvalues of a near-cone in terms of the eigenvalues of its link and deletion with respect to its apex (Proposition 7.6). We can then write down a recursive formula (Theorem 8.2) for the nonzero eigenvalues of shifted complexes, thanks to their characterization as iterated near-cones, simultaneously showing that these eigenvalues must be of the form $z(S, T)$. Finally, we independently establish a recurrence (Corollary 8.8) for the long signatures of critical pairs of a shifted complex, which coincides with the recurrence for the $z(S, T)$, thus yielding a bijection between nonzero eigenvalues and critical pairs.

Corollary 8.10 shows what the eigenvalues look like in coarse weighting. Passing from weighted to unweighted eigenvalues then easily recovers the Duval-Reiner formula for Laplacian eigenvalues of shifted complexes in terms of degree sequences [13, Thm. 1.1]. Similarly, Corollary 9.2 gives the enumeration of SST's of a shifted complex in the coarse weighting.

We are also able to show that the finely-weighted eigenvalues (though not the coarsely-weighted eigenvalues) are enough to recover the shifted complex (Corollary 8.9), or, in other words, that one can "hear the shape" of a shifted complex.

Several known results can be obtained as consequences of the general formula of Theorem 1.6.

- The complete d-skeleton of a simplex is easily seen to be shifted, and applying Theorem 1.6 to such complexes recovers Kalai's generalization of the Cayley-Prüfer Theorem.
- The one-dimensional shifted complexes are precisely the *threshold graphs*, an important class of graphs with many equivalent descriptions (see, e.g., [25]). When $d = 1$, Theorem 1.6 specializes to the weighted spanning tree enumerator for threshold graphs proved by Martin and Reiner [26, Thm. 4] and following from an independent result of Remmel and Williamson [31, Thm. 2.4].
- Thanks to an idea of Richard Ehrenborg, the formula for threshold graphs can be used to recover a theorem of Ehrenborg and van Willigenburg [14], enumerating spanning trees in certain bipartite graphs called *Ferrers graphs* (which are not in general Laplacian integral).

We discuss these corollaries in Section 10.

Some classes of complexes that we think deserve further study include *matroid complexes*, *matching complexes*, *chessboard complexes* and *color-shifted* complexes. The first three kinds of complexes are known to be Laplacian integral, by theorems of Kook, Reiner and Stanton [24], Dong and Wachs [11], and Friedman and Hanlon [17] respectively. Every matroid complex is Cohen-Macaulay [32, §III.3], hence APC, while matching complexes and chessboard complexes are APC for certain values of their defining parameters (see [6]). Color-shifted complexes, which are a common generalization of Ferrers graphs and complete colorful complexes, are not in general Laplacian integral; nevertheless, their weighted simplicial spanning tree enumerators seem to have nice factorizations.

It is our pleasure to thank Richard Ehrenborg, Vic Reiner, and Michelle Wachs for many valuable discussions. We also thank Andrew Crites and an anonymous referee for their careful reading of the manuscript.

2. Notation and definitions

2.1. **Simplicial complexes.** Let V be a finite set. A *simplicial complex on* V is a family Δ of subsets of V such that

(1) $\emptyset \in \Delta$;
(2) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The elements of V are called *vertices* of Δ, and the faces that are maximal under inclusion are called *facets*. Thus a simplicial complex is determined by its set of facets. The *dimension* of a face F is $\dim F = |F| - 1$, and the dimension of Δ is the maximum dimension of a face (or facet). The abbreviation Δ^d indicates that $\dim \Delta = d$. We say that Δ is *pure* if all facets have the same dimension; in this case, a *ridge* is a face of codimension 1, that is, dimension $\dim \Delta - 1$.

We write Δ_i for the set of i-dimensional faces of Δ, and set $f_i(\Delta) = |\Delta_i|$. The *i-skeleton* of Δ is the subcomplex of all faces of dimension $\leq i$,

$$\Delta_{(i)} = \bigcup_{-1 \leq j \leq i} \Delta_j,$$

and the *pure i-skeleton* of Δ is the subcomplex generated by the i-dimensional faces, that is,

$$\Delta_{[i]} = \{F \in \Delta \colon F \subseteq G \text{ for some } G \in \Delta_i\}.$$

We assume that the reader is familiar with simplicial homology; see, e.g., [18, §2.1]. Let Δ^d be a simplicial complex and $-1 \leq i \leq d$. Let R be a ring (if unspecified, assumed to be $\mathbb{Z}$), and let $C_i(\Delta)$ be the i^{th} simplicial chain group of Δ, i.e., the free R-module with basis $\{[F] \colon F \in \Delta_i\}$. We denote the simplicial boundary and coboundary maps respectively by

$$\partial_{\Delta,i} \ : \ C_i(\Delta) \to C_{i-1}(\Delta),$$
$$\partial^*_{\Delta,i} \ : \ C_{i-1}(\Delta) \to C_i(\Delta),$$

where we have identified cochains with chains via the natural inner product. We will abbreviate the subscripts in the notation for boundaries and coboundaries whenever no ambiguity can arise. We will often regard ∂_i (resp. ∂_i^*) as a matrix whose columns and rows (resp. rows and columns) are indexed

by Δ_i and Δ_{i-1} respectively. The i^{th} *(reduced) homology group* of Δ is $\tilde{H}_i(\Delta) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$, and the i^{th} *(reduced) Betti number* $\tilde{\beta}_i(\Delta)$ is the rank of the largest free R-module summand of $\tilde{H}_i(\Delta)$.

2.2. **Combinatorial Laplacians.** We adopt the notation of [13] for the Laplacian operators (or, equivalently, matrices) of a simplicial complex. We summarize the notation and mention some fundamental identities here.

We will often work with *multisets* (of eigenvalues or of vertices), in which each element occurs with some non-negative integer multiplicity. For brevity, we drop curly braces and commas when working with multisets of integers: for instance, 5553 denotes the multiset in which 5 occurs with multiplicity three and 3 occurs with multiplicity one. The cardinality of a multiset is the sum of the multiplicities of its elements; thus $|5553| = 4$. We write $\mathbf{a} \stackrel{\circ}{=} \mathbf{b}$ to mean that the multisets $\mathbf{a}$ and $\mathbf{b}$ differ only in their respective multiplicities of zero; for instance, $5553 \stackrel{\circ}{=} 55530 \stackrel{\circ}{=} 555300$. Of course, $\stackrel{\circ}{=}$ is an equivalence relation. The union operation $\cup$ on multisets is understood to add multiplicities: for instance, $5553 \cup 5332 = 55553332$.

For $-1 \le i \le \dim \Delta$, define linear operators $L^{\mathrm{ud}}_{\Delta,i}$, $L^{\mathrm{du}}_{\Delta,i}$, $L^{\mathrm{tot}}_{\Delta,i}$ on the vector space $C_i(\Delta)$ by

$$\begin{aligned} L^{\mathrm{ud}}_{\Delta,i} &= \partial_{i+1}\partial^*_{i+1} && (\text{the } \textit{up-down Laplacian}),\\ L^{\mathrm{du}}_{\Delta,i} &= \partial^*_i\partial_i && (\text{the } \textit{down-up Laplacian}),\\ L^{\mathrm{tot}}_{\Delta,i} &= L^{\mathrm{ud}}_{\Delta,i} + L^{\mathrm{du}}_{\Delta,i} && (\text{the } \textit{total Laplacian}). \end{aligned}$$

The *spectrum* $s^{\mathrm{tot}}_i(\Delta)$ of $L^{\mathrm{tot}}_{\Delta,i}$ is the multiset of its eigenvalues (including zero); we define $s^{\mathrm{ud}}_i(\Delta)$ and $s^{\mathrm{du}}_i(\Delta)$ similarly. Since each Laplacian operator is represented by a symmetric matrix, it is diagonalizable, so

$$|s^{\mathrm{tot}}_i(\Delta)| = |s^{\mathrm{ud}}_i(\Delta)| = |s^{\mathrm{du}}_i(\Delta)| = f_i(\Delta).$$

The various Laplacian spectra are related by the identities

$$\begin{aligned} s^{\mathrm{ud}}_i(\Delta) &\stackrel{\circ}{=} s^{\mathrm{du}}_{i+1}(\Delta),\\ s^{\mathrm{tot}}_i(\Delta) &\stackrel{\circ}{=} s^{\mathrm{ud}}_i(\Delta) \cup s^{\mathrm{du}}_i(\Delta) \end{aligned}$$

[13, eqn. (3.6)]. Therefore, each of the three families of multisets

$$\{s^{\mathrm{tot}}_i(\Delta)\colon\ -1 \le i \le \dim\Delta\}, \quad \{s^{\mathrm{ud}}_i(\Delta)\colon\ -1 \le i \le \dim\Delta\}, \quad \{s^{\mathrm{du}}_i(\Delta)\colon\ -1 \le i \le \dim\Delta\}$$

determines the other two, and we will feel free to work with whichever one is most convenient in context.

Combinatorial Laplacians and their spectra have been investigated for a number of classes of simplicial complexes. In particular, it is known that chessboard [17], matching [11], matroid [24], and shifted [13] complexes are *Laplacian integral*, i.e., all their Laplacian eigenvalues are integers. Understanding which complexes are Laplacian integral is an open question. As we will see, Laplacian eigenvalues and spanning tree enumerators are inextricably linked.

3. Simplicial spanning trees

In this section, we generalize the notion of a spanning tree to arbitrary dimension using simplicial homology, following Kalai's idea. Our definition makes sense for any ambient complex that satisfies the relatively mild APC condition.

Definition 3.1. Let Δ^d be a simplicial complex, and let $k \le d$. A *k-dimensional simplicial spanning tree* (for short, SST or k-SST) of Δ is a k-dimensional subcomplex $\Upsilon \subseteq \Delta$ such that $\Upsilon_{(k-1)} = \Delta_{(k-1)}$

and

$$\tilde{H}_k(\Upsilon) = 0, \tag{6a}$$

$$|\tilde{H}_{k-1}(\Upsilon)| < \infty, \quad \text{and} \tag{6b}$$

$$f_k(\Upsilon) = f_k(\Delta) - \tilde{\beta}_k(\Delta) + \tilde{\beta}_{k-1}(\Delta). \tag{6c}$$

We write $\mathcal{T}_k(\Delta)$ for the set of all k-SST's of Δ^d, omitting the subscript if $k = d$. Note that $\mathcal{T}_k(\Delta) = \mathcal{T}_k(\Delta_{(j)})$ for all $j \geq k$.

A zero-dimensional SST is just a vertex of Δ. If Δ is a 1-dimensional simplicial complex on n vertices—that is, a graph—then the definition of 1-SST coincides with the usual definition of a spanning tree of a graph: namely, a subgraph of Δ which is connected, acyclic, and has $n-1$ edges. Next, we give a few examples in higher dimensions.

Example 3.2. If Δ^d is a simplicial sphere (for instance, the boundary of a simplicial polytope), then deleting any facet of Δ while keeping its $(d-1)$-skeleton intact produces a d-SST. Therefore $|\mathcal{T}(\Delta)| = f_d(\Delta)$.

Example 3.3. In dimension > 1, spanning trees need not be $\mathbb{Z}$-acyclic, merely $\mathbb{Q}$-acyclic. For example, let Δ be a triangulation of the real projective plane, so that $\dim \Delta = 2$, $\tilde{H}_1(\Delta, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\tilde{H}_1(\Delta, \mathbb{Q}) = 0$. Then Δ satisfies the conditions of Definition 3.1 and is a 2-SST of itself (in fact, the only such).

Example 3.4. Consider the equatorial bipyramid B of Example 1.7. A 2-SST of B can be constructed by removing two facets F, F', provided that $F \cap F'$ contains neither of the vertices 4,5. A simple count shows that there are 15 such pairs F, F', so $|\mathcal{T}_2(B)| = 15$.

Before proceeding any further, we show that Definition 3.1 satisfies a "two-out-of-three theorem" akin to that for spanning trees of graphs.

Proposition 3.5. *Let $\Upsilon \subset \Delta^d$ be a k-dimensional subcomplex with $\Upsilon_{(k-1)} = \Delta_{(k-1)}$. Then any two of the conditions* (6a), (6b), (6c) *together imply the third.*

Proof. First, note that

$$f_\ell(\Upsilon) = f_\ell(\Delta) \quad \text{for } \ell \leq k-1 \qquad \text{and} \qquad \tilde{\beta}_\ell(\Upsilon) = \tilde{\beta}_\ell(\Delta) \quad \text{for } \ell \leq k-2. \tag{7}$$

Next, we use the standard fact that the Euler characteristic $\chi(\Upsilon)$ can be calculated as the alternating sum either of the f-numbers or of the Betti numbers. Thus

$$\begin{aligned} \chi(\Upsilon) &= \sum_{i=0}^{k} (-1)^i f_i(\Upsilon) \\ &= (-1)^k f_k(\Upsilon) + \sum_{i=0}^{k-1} (-1)^i f_i(\Delta) \\ &= (-1)^k f_k(\Upsilon) + \chi(\Delta) - (-1)^k f_k(\Delta) \end{aligned} \tag{8}$$

and on the other hand,

$$\begin{aligned} \chi(\Upsilon) &= \sum_{i=0}^{k} (-1)^i \tilde{\beta}_i(\Upsilon) \\ &= (-1)^k (\tilde{\beta}_k(\Upsilon) - \tilde{\beta}_{k-1}(\Upsilon)) + \sum_{i=0}^{k-2} (-1)^i \tilde{\beta}_i(\Delta) \\ &= (-1)^k (\tilde{\beta}_k(\Upsilon) - \tilde{\beta}_{k-1}(\Upsilon)) + \chi(\Delta) - (-1)^k (\tilde{\beta}_k(\Delta) - \tilde{\beta}_{k-1}(\Delta)). \end{aligned} \tag{9}$$

Equating (8) and (9) gives

$$f_k(\Upsilon) - f_k(\Delta) = \tilde{\beta}_k(\Upsilon) - \tilde{\beta}_{k-1}(\Upsilon) - \tilde{\beta}_k(\Delta) + \tilde{\beta}_{k-1}(\Delta)$$

or equivalently

$$\big(f_k(\Upsilon) - f_k(\Delta) + \tilde{\beta}_k(\Delta) - \tilde{\beta}_{k-1}(\Delta)\big) - \tilde{\beta}_k(\Upsilon) + \tilde{\beta}_{k-1}(\Upsilon) = 0.$$

Since (6a) says that $\tilde{\beta}_k(\Upsilon) = 0$ (note that $\tilde{H}_k(\Upsilon)$ must be free abelian) and (6b) says that $\tilde{\beta}_{k-1}(\Upsilon) = 0$, the conclusion follows. □

Definition 3.6. A simplicial complex Δ^d is *acyclic in positive codimension*, or APC for short, if $\tilde{\beta}_j(\Delta) = 0$ for all $j < d$.

Equivalently, a complex Δ^d is APC if it has the homology type of a wedge of zero or more d-dimensional spheres. In particular, any Cohen-Macaulay complex is APC. The converse is very far from true, because, for instance, an APC complex need not even be pure. For our purposes, the APC complexes are the "correct" simplicial analogues of connected graphs for the following reason.

Proposition 3.7. *For any simplicial complex Δ^d, the following are equivalent:*

(1) *Δ is APC.*
(2) *Δ has a d-dimensional spanning tree.*
(3) *Δ has a k-dimensional spanning tree for every $k \le d$.*

Proof. It is trivial that (3) implies (2). To see that (2) implies (1), suppose that Δ has a d-dimensional spanning tree Υ. Then $\Upsilon_i = \Delta_i$ for all $i \le d-1$, so $\tilde{H}_i(\Delta) = \tilde{H}_i(\Upsilon) = 0$ for all $i \le d-2$. Moreover, in the diagram

$$\begin{array}{ccccc} C_d(\Delta) & \xrightarrow{\partial_{\Delta,d}} & C_{d-1}(\Delta) & \xrightarrow{\partial_{\Delta,d-1}} & C_{d-2}(\Delta) \\ \bigcup & & \| & & \| \\ C_d(\Upsilon) & \xrightarrow{\partial_{\Upsilon,d}} & C_{d-1}(\Upsilon) & \xrightarrow{\partial_{\Upsilon,d-1}} & C_{d-2}(\Upsilon) \end{array}$$

we have $\ker \partial_{\Delta,d-1} = \ker \partial_{\Upsilon,d-1}$ and $\operatorname{im} \partial_{\Delta,d} \supseteq \operatorname{im} \partial_{\Upsilon,d}$, so there is a surjection $0 = \tilde{H}_{d-1}(\Upsilon) \to \tilde{H}_{d-1}(\Delta)$, implying that Δ is APC.

To prove (1) implies (3), it suffices to consider the case $k = d$, because any skeleton of an APC complex is also APC. We can construct a d-SST Υ by the following algorithm. Let $\Upsilon = \Delta$. If $\tilde{H}_d(\Upsilon) \neq 0$, then there is some nonzero linear combination of facets of Υ that is mapped to zero by $\partial_{\Upsilon,d}$. Let F be one of those facets, and let $\Upsilon' = \Upsilon \backslash \{F\}$. Then $\tilde{\beta}_d(\Upsilon') = \tilde{\beta}_d(\Upsilon) - 1$ and $\tilde{\beta}_i(\Upsilon') = \tilde{\beta}_i(\Upsilon)$ for $i \le d-2$, and by the Euler characteristic formula, we have $\tilde{\beta}_{d-1}(\Upsilon') = \tilde{\beta}_{d-1}(\Upsilon)$ as well. Replacing Υ with Υ' and repeating, we eventually arrive at the case $\tilde{H}_d(\Upsilon) = 0$, when Υ is a d-SST of Δ. □

The APC condition is a fairly mild one. For instance, any $\mathbb{Q}$-acyclic complex is clearly APC (and is its own unique SST), as is any Cohen-Macaulay complex (in particular, any shifted complex).

4. Simplicial analogues of the Matrix-Tree Theorem

We now explain how to enumerate simplicial spanning trees of a complex using its Laplacian. Throughout this section, let Δ^d be an APC simplicial complex on vertex set $[n]$. For $k \le d$, define

$$\pi_k = \pi_k(\Delta) = \prod_{0 \neq \lambda \in s^{\mathrm{ud}}_{k-1}(\Delta)} \lambda, \qquad \tau_k = \tau_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon)|^2.$$

We are interested in the relationships between these two families of invariants. When $d = 1$, the relationship is given by Theorem 1.1. In the notation just defined, part (1) of that theorem says that $\tau_1 = \pi_1/n$, and part (2) says that $\tau_1 = \det L_i$ (i.e., the determinant of the reduced Laplacian obtained from $L^{\mathrm{ud}}_{\Delta,0}$ by deleting the row and column corresponding to any vertex i).

The results of this section generalize both parts of the Matrix-Tree Theorem from graphs to all APC complexes Δ^d. Our arguments are closely based on those used by Kalai [20] and Adin [1] to enumerate SST's of skeletons of simplices and of complete colorful complexes.

We begin by setting up some notation. Abbreviate $\tilde{\beta}_i = \tilde{\beta}_i(\Delta)$, $f_i = f_i(\Delta)$, and $\partial = \partial_{\Delta,d}$. Let T be a set of facets of Δ of cardinality $f_d - \tilde{\beta}_d + \tilde{\beta}_{d-1} = f_d - \tilde{\beta}_d$, and let S be a set of ridges such that $|S| = |T|$. Define

$$\Delta_T = T \cup \Delta_{(d-1)}, \qquad \bar{S} = \Delta_{(d-1)} \setminus S, \qquad \Delta_{\bar{S}} = \bar{S} \cup \Delta_{(d-2)},$$

and let $\partial_{S,T}$ be the square submatrix of ∂ with rows indexed by S and columns indexed by T.

Proposition 4.1. *The matrix $\partial_{S,T}$ is nonsingular if and only if $\Delta_T \in \mathcal{T}_d(\Delta)$ and $\Delta_{\bar{S}} \in \mathcal{T}_{d-1}(\Delta)$.*

Proof. We may regard $\partial_{S,T}$ as the top boundary map of the d-dimensional relative complex $\Gamma = (\Delta_T, \Delta_{\bar{S}})$. So $\partial_{S,T}$ is nonsingular if and only if $\tilde{H}_d(\Gamma) = 0$. Consider the long exact sequence

$$0 \to \tilde{H}_d(\Delta_{\bar{S}}) \to \tilde{H}_d(\Delta_T) \to \tilde{H}_d(\Gamma) \to \tilde{H}_{d-1}(\Delta_{\bar{S}}) \to \tilde{H}_{d-1}(\Delta_T) \to \tilde{H}_{d-1}(\Gamma) \to \cdots \tag{10}$$

If $\tilde{H}_d(\Gamma) \neq 0$, then $\tilde{H}_d(\Delta_T)$ and $\tilde{H}_{d-1}(\Delta_{\bar{S}})$ cannot both be zero. This proves the "only if" direction.

If $\tilde{H}_d(\Gamma) = 0$, then $\tilde{H}_d(\Delta_{\bar{S}}) = 0$ (since $\dim \Delta_{\bar{S}} = d-1$), so (10) implies $\tilde{H}_d(\Delta_T) = 0$. Therefore Δ_T is a d-tree, because it has the correct number of facets. Hence $\tilde{H}_{d-1}(\Delta_T)$ is finite. Then (10) implies that $\tilde{H}_{d-1}(\Delta_{\bar{S}})$ is finite. In fact, it is zero because the top homology group of any complex must be torsion-free. Meanwhile, $\Delta_{\bar{S}}$ has the correct number of facets to be a $(d-1)$-SST of Δ, proving the "if" direction. □

Proposition 4.2. *If $\partial_{S,T}$ is nonsingular, then*

$$|\det \partial_{S,T}| = \frac{|\tilde{H}_{d-1}(\Delta_T)| \cdot |\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta_T)|} = \frac{|\tilde{H}_{d-1}(\Delta_T)| \cdot |\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta)|}.$$

Proof. As before, we interpret $\partial_{S,T}$ as the boundary map of the relative complex $\Gamma = (\Delta_T, \Delta_{\bar{S}})$. So $\partial_{S,T}$ is a map from $\mathbb{Z}^{|T|}$ to $\mathbb{Z}^{|T|}$, and $\mathbb{Z}^{|T|}/\partial_{S,T}(\mathbb{Z}^{|T|})$ is a finite abelian group of order $|\det \partial_{S,T}|$. On the other hand, since Γ has no faces of dimension $\leq d-2$, its lower boundary maps are all zero, so $|\det \partial_{S,T}| = |\tilde{H}_{d-1}(\Gamma)|$. Since $\tilde{H}_{d-2}(\Delta_T)$ is finite, the desired result now follows from the piece

$$0 \to \tilde{H}_{d-1}(\Delta_T) \to \tilde{H}_{d-1}(\Gamma) \to \tilde{H}_{d-2}(\Delta_{\bar{S}}) \to \tilde{H}_{d-2}(\Delta_T) \to 0 \tag{11}$$

of the long exact sequence (10). □

We can now prove the first version of the Simplicial Matrix-Tree Theorem, relating the quantities π_d and τ_d. Abbreviate $L = L^{\mathrm{ud}}_{\Delta,d-1}$.

Theorem 1.3 (Simplicial Matrix-Tree Theorem). *Let Δ^d be an APC simplicial complex. Then:*

(1) *We have*

$$\pi_d(\Delta) = \frac{\tau_d(\Delta)\tau_{d-1}(\Delta)}{|\tilde{H}_{d-2}(\Delta)|^2}.$$

(2) *Let U be the set of facets of a $(d-1)$-SST of Δ, and let L_U denote the reduced Laplacian*[3] *obtained by deleting the rows and columns of L corresponding to U. Then*

$$\tau_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Delta_U)|^2} \det L_U.$$

[3]A warning: This notation for reduced Laplacians specifies which rows and columns to *exclude* (in analogy to the notation L_i in the statement of Theorem 1.1), in contrast to the notation $\partial_{S,T}$ for restricted boundary maps, which specifies which rows and columns to *include*.

Proof of Theorem 1.3 (1). The Laplacian L is a square matrix with f_{d-1} rows and columns, and rank $f_d - \tilde{\beta}_d = f_d - \tilde{\beta}_d + \tilde{\beta}_{d-1}$ (because Δ is APC). Let $\chi(L;y) = \det(yI - L)$ be its characteristic polynomial (where I is an identity matrix), so that $\pi_d(\Delta)$, the product of the nonzero eigenvalues of L, is given (up to sign) by the coefficient of $y^{f_{d-1}-f_d+\tilde{\beta}_d}$ in $\chi(L;y)$. Equivalently,

$$\pi_d \;=\; \sum_{\substack{S\subset\Delta_{d-1}\\ |S|=\operatorname{rank} L}} \det L_U \;=\; \sum_{\substack{S\subset\Delta_{d-1}\\ |S|=f_d-\tilde{\beta}_d}} \det L_U \tag{12}$$

where $U = \Delta_{d-1}\backslash S$ in each summand. By the Binet-Cauchy formula, we have

$$\det L_U \;=\; \sum_{\substack{T\subset\Delta_d\\ |T|=|S|}} (\det \partial_{S,T})(\det \partial^*_{S,T}) \;=\; \sum_{\substack{T\subset\Delta_d\\ |T|=|S|}} (\det \partial_{S,T})^2. \tag{13}$$

Combining (12) and (13), applying Proposition 4.1, and interchanging the sums, we obtain

$$\pi_d = \sum_{T:\Delta_T\in\mathcal{T}_d(\Delta)} \quad \sum_{S:\Delta_{\bar{S}}\in\mathcal{T}_{d-1}(\Delta)} (\det \partial_{S,T})^2$$

and now applying Proposition 4.2 yields

$$\pi_d = \sum_{T:\Delta_T\in\mathcal{T}_d(\Delta)} \quad \sum_{S:\Delta_{\bar{S}}\in\mathcal{T}_{d-1}(\Delta)} \left(\frac{|\tilde{H}_{d-1}(\Delta_T)|\cdot|\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta)|}\right)^2$$

$$= \frac{\left(\displaystyle\sum_{T:\Delta_T\in\mathcal{T}_d(\Delta)} |\tilde{H}_{d-1}(\Delta_T)|^2\right)\left(\displaystyle\sum_{S:\Delta_{\bar{S}}\in\mathcal{T}_{d-1}(\Delta)} |\tilde{H}_{d-2}(\Delta_{\bar{S}})|\right)}{|\tilde{H}_{d-2}(\Delta)|^2}$$

as desired. □

In order to prove the "reduced Laplacian" part of Theorem 1.3, we first check that when we delete the rows of ∂ corresponding to a $(d-1)$-SST, the resulting reduced Laplacian has the correct size, namely, that of a d-SST.

Lemma 4.3. *Let U be the set of facets of a $(d-1)$-SST of Δ, and let $S = \Delta_{d-1}\backslash U$. Then $|S| = f_d(\Delta) - \tilde{\beta}_d(\Delta)$, the number of facets of a d-SST of Δ.*

Proof. Let $\Gamma = \Delta_{(d-1)}$. By Proposition 3.5 and the observation (7), $|U| = f_{d-1}(\Gamma) - \tilde{\beta}_{d-1}(\Gamma) + \tilde{\beta}_{d-2}(\Gamma) = f_{d-1}(\Delta) - \tilde{\beta}_{d-1}(\Gamma)$, so $|S| = \tilde{\beta}_{d-1}(\Gamma)$. The Euler characteristics of Δ and Γ are

$$\chi(\Delta) = \sum_{i=0}^{d}(-1)^i f_i(\Delta) = \sum_{i=0}^{d}(-1)^i \tilde{\beta}_i(\Delta),$$
$$\chi(\Gamma) = \sum_{i=0}^{d-1}(-1)^i f_i(\Gamma) = \sum_{i=0}^{d-1}(-1)^i \tilde{\beta}_i(\Gamma).$$

By (7), we see that

$$\chi(\Delta) - \chi(\Gamma) \;=\; (-1)^d f_d(\Delta) \;=\; (-1)^d\tilde{\beta}_d(\Delta) + (-1)^{d-1}\tilde{\beta}_{d-1}(\Delta) - (-1)^{d-1}\tilde{\beta}_{d-1}(\Gamma)$$

from which we obtain $f_d(\Delta) = \tilde{\beta}_d(\Delta) - \tilde{\beta}_{d-1}(\Delta) + \tilde{\beta}_{d-1}(\Gamma)$. Since Δ is APC, we have $\tilde{\beta}_{d-1}(\Delta) = 0$, so $|S| = \tilde{\beta}_{d-1}(\Gamma) = f_d(\Delta) - \tilde{\beta}_d(\Delta)$ as desired. □

Proof of Theorem 1.3 (2). By the Binet-Cauchy formula, we have

$$\det L_U = \sum_{T:\ |T|=|S|} (\det \partial_{S,T})(\det \partial^*_{S,T}) = \sum_{T:\ |T|=|S|} (\det \partial_{S,T})^2.$$

By Lemma 4.3 and Proposition 4.1, $\partial_{S,T}$ is nonsingular exactly when $\Delta_T \in \mathcal{T}_d(\Delta)$. Hence Proposition 4.2 gives

$$\begin{aligned}\det L_U &= \sum_{T:\Delta_T\in\mathcal{T}_d(\Delta)} \left(\frac{|\tilde{H}_{d-1}(\Delta_T)|\cdot|\tilde{H}_{d-2}(\Delta_U)|}{|\tilde{H}_{d-2}(\Delta)|}\right)^2 \\ &= \frac{|\tilde{H}_{d-2}(\Delta_U)|^2}{|\tilde{H}_{d-2}(\Delta)|^2} \sum_{T:\Delta_T\in\mathcal{T}_d(\Delta)} |\tilde{H}_{d-1}(\Delta_T)|^2 \ = \ \frac{|\tilde{H}_{d-2}(\Delta_U)|^2}{|\tilde{H}_{d-2}(\Delta)|^2}\tau_d(\Delta),\end{aligned}$$

which is equivalent to the desired formula. □

Remark 4.4. Suppose that $\tilde{H}_{d-2}(\Delta) = 0$ (for example, if Δ is Cohen-Macaulay). Then the two versions of Theorem 1.3 assert that

$$\tau_d = \frac{\pi_d}{\tau_{d-1}} = \frac{\det L_U}{|\tilde{H}_{d-2}(\Delta_U)|^2},$$

from which it is easy to recognize the two different versions of the classical Matrix-Tree Theorem, Theorem 1.1. (A graph is Cohen-Macaulay as a simplicial complex if and only if it is connected.) Moreover, the recurrence $\tau_d = \pi_d/\tau_{d-1}$ leads to an expression for τ_d as an alternating product of eigenvalues:

$$\tau_d = \frac{\pi_d\pi_{d-2}\cdots}{\pi_{d-1}\pi_{d-3}\cdots} = \prod_{k=0}^{d} \pi_k^{(-1)^{d-k}}. \tag{14}$$

This formula is reminiscent of the Reidemeister torsion of a chain complex or CW-complex (although τ_d is of course not a topological invariant); see, e.g., [35]. Furthermore, (14) is in practice an efficient way to calculate τ_d.

Example 4.5. For the equatorial bipyramid B, we have

$$\pi_0(B) = 5, \qquad \pi_1(B) = 5\cdot 5\cdot 5\cdot 3 = 375, \qquad \pi_2(B) = 5\cdot 5\cdot 5\cdot 3\cdot 3 = 1125.$$

These numbers can be checked by computation, and also follow from the Duval-Reiner formula for Laplacian eigenvalues of a shifted complex. Applying the alternating product formula (14) yields

$$\tau_0(B) = 5, \qquad \tau_1(B) = 375/5 = 75, \qquad \tau_2(B) = \frac{1125\cdot 5}{375} = 15.$$

Indeed, $\tau_0(B)$ is the number of vertices. Cayley's formula implies that deleting any one edge e from K_n yields a graph with $(n-2)n^{n-3}$ spanning trees (because e itself belongs to $(n-1)/\binom{n}{2}$ of the spanning trees of K_n), and the 1-skeleton $B_{(1)}$ is such a graph with $n = 5$, so $\tau_1(B) = 75$. Finally, we have seen in Example 3.4 that $\tau_2(B) = 15$.

5. Weighted enumeration of simplicial spanning trees

We can obtain much finer enumerative information by labeling the facets of a complex with indeterminates, so that the invariant τ_k becomes a generating function for its SST's.

Let Δ^d be an APC simplicial complex, and let $\partial = \partial_{\Delta,d}$. Introduce an indeterminate x_F for each facet F of maximum dimension, and let $X_F = x_F^2$. For every $T \subseteq \Delta_d$, let $x_T = \prod_{F\in T} x_F$ and let $X_T = x_T^2$. To construct the *weighted boundary matrix* $\hat{\partial}$ from ∂, multiply each column of $\hat{\partial}$ by x_F, where F is the facet of Δ corresponding to that column. The *weighted coboundary* $\hat{\partial}^*$ is the transpose of $\hat{\partial}$. We can now define weighted versions of Laplacians, the various submatrices of the boundary and coboundary matrices used in Section 4, and the invariants π_k and τ_k. We will notate

each weighted invariant by placing a hat over the symbol for the corresponding unweighted quantity. Thus $\hat{\pi}_k$ is the product of the nonzero eigenvalues of $\hat{L}^{\mathrm{ud}}_{\Delta,k-1}$, and

$$\hat{\tau}_k = \hat{\tau}_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}_k(\Delta)} |\tilde{H}_{k-1}(\Upsilon)|^2 X_\Upsilon.$$

To recover any unweighted quantity from its weighted analogue, set $x_F = 1$ for all $F \in \Delta_d$.

Proposition 5.1. *Let $T \subset \Delta_d$ and $S \subset \Delta_{d-1}$, with $|T| = |S| = f_d - \tilde{\beta}_d$. Then $\det \hat{\partial}_{S,T} = x_T \det \partial_{S,T}$ is nonzero if and only if $\Delta_T \in \mathcal{T}_d(\Delta)$ and $\Delta_{\bar{S}} \in \mathcal{T}_{d-1}(\Delta)$. In that case,*

$$\pm \det \hat{\partial}_{S,T} = \frac{|\tilde{H}_{d-1}(\Delta_T)| \cdot |\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta_T)|} x_T = \frac{|\tilde{H}_{d-1}(\Delta_T)| \cdot |\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta)|} x_T. \tag{15}$$

Proof. The first claim follows from Proposition 4.1, and the second from Proposition 4.2. □

It is now straightforward to adapt the proofs of both parts of Theorem 1.3 to the weighted setting. For convenience, we restate the result. Let $\hat{L} = \hat{L}^{\mathrm{ud}}_{\Delta,d-1}$.

Theorem 1.4 (**Weighted Simplicial Matrix-Tree Theorem**). *Let Δ^d be an APC simplicial complex. Then:*

(1) *We have*

$$\hat{\pi}_d(\Delta) = \frac{\hat{\tau}_d(\Delta)\tau_{d-1}(\Delta)}{|\tilde{H}_{d-2}(\Delta)|^2}.$$

(2) *Let U be the set of facets of a $(d-1)$-SST of Δ, and let $\hat{L}_U$ be the reduced Laplacian obtained by deleting the rows and columns of $\hat{L}$ corresponding to U. Then*

$$\hat{\tau}_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Delta_U)|^2} \det \hat{L}_U.$$

Proof. For assertion (1), we use a weighted version of the argument of part (1) of Theorem 1.3. By the Binet-Cauchy formula and Proposition 5.1, we have

$$\hat{\pi}_d = \sum_{S \subset \Delta_{d-1}} \sum_{\substack{T \subset \Delta_d \\ |T|=|S|}} (\det \hat{\partial}^*_{T,S})(\det \hat{\partial}_{S,T}) = \sum_S \sum_T (\det \hat{\partial}_{S,T})^2$$

$$= \sum_{T:\Delta_T \in \mathcal{T}_d(\Delta)} \sum_{S:\Delta_{\bar{S}} \in \mathcal{T}_{d-1}(\Delta)} (\det \hat{\partial}_{S,T})^2$$

$$= \sum_{T:\Delta_T \in \mathcal{T}_d(\Delta)} \sum_{S:\Delta_{\bar{S}} \in \mathcal{T}_{d-1}(\Delta)} \left(\frac{|\tilde{H}_{d-1}(\Delta_T)| \cdot |\tilde{H}_{d-2}(\Delta_{\bar{S}})|}{|\tilde{H}_{d-2}(\Delta)|} \right)^2 X_T = \frac{\hat{\tau}_d(\Delta)\tau_{d-1}(\Delta)}{|\tilde{H}_{d-2}(\Delta)|^2}.$$

The proof of assertion (2) of the theorem is identical to that of part (2) of Theorem 1.3, using Proposition 5.1 instead of Proposition 4.2. □

Example 5.2. We return to the equatorial bipyramid B of Example 1.7. Weight each facet $F = \{i, j, k\}$ by the monomial $x_F = x_i x_j x_k$. Let $U = \{12, 13, 14, 15\}$ be the facets of a 1-SST of $B_{(1)}$. Then the reduced Laplacian $\hat{L}_U$ is

$$\begin{pmatrix} x_2x_3(x_1+x_4+x_5) & -x_2x_3x_5 & x_2x_3x_4 & x_2x_3x_5 & -x_2x_3x_4 \\ -x_2x_3x_4 & x_2x_5(x_1+x_3) & 0 & -x_2x_3x_5 & 0 \\ x_2x_3x_4 & 0 & x_3x_4(x_1+x_2) & 0 & -x_2x_3x_4 \\ x_2x_3x_5 & -x_2x_3x_5 & 0 & x_3x_5(x_1+x_2) & 0 \\ -x_2x_3x_4 & 0 & -x_2x_3x_4 & 0 & x_2x_4(x_1+x_3) \end{pmatrix}$$

and the generating function for 2-SST's by their degree sequences is

$$\hat{\tau}_2(B) = \det \hat{L}_S = \sum_{\Upsilon \in \mathcal{T}(B)} \prod_{i \in [5]} x_i^{\deg_B(i)} = x_1^3 x_2^3 x_3^3 x_4^2 x_5^2 (x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)$$

where $\deg_B(i)$ means the number of facets of B containing vertex i. Setting $x_i = 1$ for every i recovers the unweighted equality $\tau_2(B) = 15$ (see Examples 3.4 and 4.5).

6. Shifted complexes

6.1. **General definitions.** In the next several sections of the paper, we apply the tools just developed to the important class of *shifted complexes*. We begin by reviewing some standard facts about shifted complexes and shifted families; for more details, see, e.g., [21].

Let k be an integer. A *k-set* is a set of integers of cardinality k. A *k-family* is a set of k-sets (for example, the set of $(k-1)$-dimensional faces of a simplicial complex). The *componentwise partial order* $\preceq$ on k-sets of integers is defined as follows: if $A = \{a_1 < a_2 < \cdots < a_k\}$ and $B = \{b_1 < b_2 < \cdots < b_k\}$, then $A \preceq B$ if $a_j \leq b_j$ for all j. A k-family $\mathcal{F}$ is *shifted* if $B \in \mathcal{F}$ and $A \preceq B$ together imply $A \in \mathcal{F}$. Equivalently, $\mathcal{F}$ is shifted if it is an order ideal with respect to the componentwise partial order. A simplicial complex Σ is *shifted* if Σ_i is shifted for all i. Accordingly, we may specify a shifted complex by the list of its facets that are maximal with respect to $\preceq$, writing $\Sigma = \langle F_1, \ldots, F_n \rangle$. For example, the bipyramid of Example 1.7 is the shifted complex $\langle 235 \rangle$. We will not lose any generality by assuming that the vertex set for every shifted complex we encounter is an integer interval $[p, q] = \{p, p+1, \ldots, q\}$; in particular, we will use the symbol p throughout for the vertex with the smallest index.

The *deletion* and *link* of Σ with respect to p are defined to be the subcomplexes

$$\Delta = \operatorname{del}_p \Sigma = \{F \backslash \{p\} \colon F \in \Sigma\},$$
$$\Lambda = \operatorname{link}_p \Sigma = \{F \colon p \notin F,\ F \cup \{p\} \in \Sigma\}.$$

It is easy to see that the deletion and link of a shifted complex on $[p, q]$ are themselves shifted complexes[4] on $[p+1, q]$.

A complex Σ on vertex set V is called a *near-cone with apex p* if it has the following property: if $F \in \operatorname{del}_p \Sigma$ and $v \in F$, then $F \backslash \{v\} \in \operatorname{link}_p \Sigma$ (equivalently, $F \backslash \{v\} \cup \{p\} \in \Sigma$). It is easy to see that a shifted complex on $[p, q]$ is a near-cone with apex p. Björner and Kalai [5, Theorem 4.3] showed that the Betti numbers of a shifted complex Σ (indeed, of a near-cone) with initial vertex p are given by

$$\tilde{\beta}_i(\Sigma) = |\{F \in \Sigma_i \colon p \notin F,\ F \dot{\cup} \{p\} \notin \Sigma\}|. \tag{16}$$

6.2. **The combinatorial fine weighting.** Let $\{x_{i,j}\}$ be a set of indeterminates, indexed by integers i, j. Let $\Bbbk$ be the field of rational functions in the $x_{i,j}$ with coefficients in $\mathbb{C}$ (or in any other field of characteristic zero). Since these indeterminates will often appear squared, we set $X_{i,j} = x_{i,j}^2$. The *combinatorial fine weighting* assigns to a multiset of vertices $S = \{i_1 \leq i_2 \leq \cdots \leq i_m\}$ the monomials

$$x_S = x_{1,i_1} x_{2,i_2} \cdots x_{m,i_m} \quad \text{and} \quad X_S = X_{1,i_1} X_{2,i_2} \cdots X_{m,i_m}. \tag{17}$$

Our goal is to describe the generating function

$$\hat{\tau}_d(\Sigma) = \sum_{\Upsilon \in \mathcal{T}(\Sigma)} |\tilde{H}_{d-1}(\Sigma, \mathbb{Z})|^2 X_\Upsilon$$

of a shifted complex Σ, where, for each simplicial spanning tree Υ, the monomial

$$X_\Upsilon = \prod_{\text{facets } F \in \Upsilon} X_F$$

[4]This is also true for the deletion and link with respect to any vertex, not just p, but then the resulting vertex set is no longer a set of consecutive integers. Since we will not have any need to take the deletion and link with respect to any vertex other than p, we won't worry about that, and instead enjoy the resulting simplicity of specifying the new minimal vertex of the deletion and link.

records both the number of facets of Υ containing each vertex of Σ, as well as the order in which the vertices appear in facets.

Define the "raising operator" $\uparrow$ by $\uparrow x_{i,j} = x_{i+1,j}$ for $i \le d$ and $\uparrow x_{d+1,j} = 0$. We extend $\uparrow$ linearly and multiplicatively to an operator on all of $\Bbbk$. The raising operator can also be applied to a $\Bbbk$-linear operator f by the rule

$$(\uparrow f)(V) = \uparrow(f(\uparrow^{-1}(V))) \tag{18}$$

for any vector V over $\Bbbk$. The a^{th} iterate of $\uparrow$ is denoted $\uparrow^a$.

The following identities will be useful. Let $\tilde{S} = S \cup \{p\}$, where $\cup$ denotes the union as multisets, so that the multiplicity of p in $\tilde{S}$ is one more than its multiplicity in S. Then, for all integers a, j,

$$\uparrow^a x_{1,p} \cdot \uparrow^{a+1} x_{S \cup j} \;= \uparrow^a \big(x_{1,p} \cdot \uparrow x_{S \cup j}\big) \;= \uparrow^a x_{\tilde{S} \cup j} \tag{19a}$$

and

$$\frac{\uparrow^a x_{\tilde{S}}}{\uparrow^{a+1} x_S} \;=\; x_{a+1,p} \;= \uparrow^a x_{1,p}. \tag{19b}$$

The same identities hold if x is replaced with X.

Now, define the *combinatorially finely weighted simplicial boundary map* of Σ as the homomorphism $\hat{\partial} = \hat{\partial}_{\Sigma,i} : C_i(\Sigma) \to C_{i-1}(\Sigma)$ which acts on generators $[F]$ (for $F \in \Sigma_i$) by

$$\hat{\partial}[F] = \sum_{j \in F} \varepsilon(j, F) \uparrow^{d-i} x_F \, [F \backslash j]. \tag{20}$$

Here we have set $\varepsilon(v, F) = (-1)^{j+1}$ if v is the j^{th} smallest vertex of F, and $\varepsilon(v, F) = 0$ if $v \notin F$. Similarly define the *finely weighted simplicial coboundary map* $\hat{\partial}^* = \hat{\partial}^*_{F,i+1} : C_i(\Sigma) \to C_{i+1}(\Sigma)$ by

$$\hat{\partial}^*[F] = \sum_{j \in V \setminus F} \varepsilon(j, F \cup j) \uparrow^{d-i-1} x_{F \cup j} \, [F \cup j]. \tag{21}$$

These maps do *not* make the chain groups of Σ into an algebraic chain complex, because $\hat{\partial}\hat{\partial}$ and $\hat{\partial}^*\hat{\partial}^*$ do not vanish in general. (We will fix this problem in Section 6.3.) On the other hand, they have combinatorial significance, because we will be able to apply part (2) of Theorem 1.4 to the *finely weighted up-down Laplacian* $\hat{L}^{\mathrm{ud}} = \hat{\partial}_d \hat{\partial}_d^*$. This Laplacian may be regarded as a matrix whose rows and columns are indexed by Σ_{d-1}. It is not hard to check that for each $F, G \in \Sigma_{d-1}$, the corresponding entry of $\hat{L}^{\mathrm{ud}}$ is

$$(\hat{L}^{\mathrm{ud}})_{FG} = \begin{cases} \varepsilon(j, H)\, \varepsilon(i, H)\, X_H & \text{if } H = F \cup j = G \cup i \in \Sigma, \\ \displaystyle\sum_{j \colon F \cup j \in \Sigma} X_{F \cup j} & \text{if } F = G, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Let U be the simplicial spanning tree of $\Sigma_{(d-1)}$ consisting of all ridges containing vertex p. (This subcomplex is an SST because it has a complete $(d-2)$-skeleton and is a cone over p, hence contractible.) Let $\hat{L}^{\mathrm{ud}}_U$ be the reduced Laplacian obtained from $\hat{L}^{\mathrm{ud}}$ by deleting the corresponding rows and columns, so that the remaining rows and columns are indexed by the facets of $\Lambda = \mathrm{link}_p \Sigma$. Then part (2) of Theorem 1.4 asserts that $\hat{\tau}_d(\Sigma) = \det \hat{L}^{\mathrm{ud}}_U$.

Let $N = N(\Sigma)$ be the matrix obtained from $\hat{L}^{\mathrm{ud}}_U$ by dividing each row F by $\uparrow x_F$ and dividing each column G by $\uparrow x_G$. The (F, G) entry of N is thus

$$N_{FG} = \begin{cases} \varepsilon(j, H)\, \varepsilon(i, H)\, \dfrac{X_H}{\uparrow x_F \;\; \uparrow x_G} & \text{if } H = F \cup j = G \cup i \in \Sigma, \\ \displaystyle\sum_{j \colon F \cup j \in \Sigma} \frac{X_{F \cup j}}{\uparrow X_F} & \text{if } F = G, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Moreover,

$$\hat{\tau}_d(\Sigma) = \det \hat{L}_U^{\mathrm{ud}} = \left(\prod_{F \in \Lambda_{d-1}} \uparrow X_F \right) \det N. \tag{24}$$

We will shortly see (Lemma 6.3) that N is almost identical to the (full) Laplacian of the deletion $\Delta = \mathrm{del}_p \Sigma$.

6.3. **The algebraic fine weighting.** The combinatorial fine weighting just defined is awkward to work with directly, because the simplicial boundary and coboundary maps (20) and (21) do not fit together into an algebraic chain complex. Therefore, we introduce a new weighting by Laurent monomials, the *algebraic fine weighting*, that does give the structure of a chain complex, behaves well with respect to cones and near-cones, and is easy to translate into the combinatorial fine weighting.

Definition 6.1. Let Δ^d be a simplicial complex on vertices $V \subset \mathbb{N}$. The *algebraic finely weighted simplicial boundary map* of Δ is the homomorphism $\boldsymbol{\partial}_{\Delta,i} : C_i(\Delta) \to C_{i-1}(\Delta)$ given by

$$\boldsymbol{\partial}_{\Delta,i}[F] = \sum_{j \in F} \varepsilon(j,F) \frac{\uparrow^{d-i}(x_F)}{\uparrow^{d-i+1}(x_{F\setminus j})} [F \setminus j] \tag{25}$$

and similarly the *algebraic finely weighted simplicial coboundary map* $\boldsymbol{\partial}^*_{\Delta,i+1} : C_i(\Delta) \to C_{i+1}(\Delta)$ is given by

$$\boldsymbol{\partial}^*_{\Delta,i+1}[F] = \sum_{j \in V \setminus F} \varepsilon(j, F \cup j) \frac{\uparrow^{d-i-1}(x_{F \cup j})}{\uparrow^{d-i}(x_F)} [F \cup j]. \tag{26}$$

We will sometimes drop one or both subscripts when no confusion can arise. By the formula (18), we can apply the raising operator $\uparrow$to $\boldsymbol{\partial}$ and $\boldsymbol{\partial}^*$ by applying it to each matrix entry. That is, for $[F] \in C_i(\Delta)$ and $a \in \mathbb{N}$, we have

$$\uparrow^a \boldsymbol{\partial}_{\Delta,i}[F] = \sum_{j \in F} \varepsilon(j,F) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+1}(x_{F\setminus j})} [F \setminus j], \tag{27}$$

$$\uparrow^a \boldsymbol{\partial}^*_{\Delta,i}[F] = \sum_{j \in V \setminus F} \varepsilon(j, F \cup j) \frac{\uparrow^{d-i+a-1}(x_{F \cup j})}{\uparrow^{d-i+a}(x_F)} [F \cup j]. \tag{28}$$

Lemma 6.2. *Let $a \in \mathbb{N}$. Then $\uparrow^a \boldsymbol{\partial} \circ \uparrow^a \boldsymbol{\partial} = 0$ and $\uparrow^a \boldsymbol{\partial}^* \circ \uparrow^a \boldsymbol{\partial}^* = 0$. That is, the algebraic finely weighted boundary and coboundary operators induce chain complexes*

$$\cdots \to C_{i+1}(\Delta) \xrightarrow{\boldsymbol{\partial}_{\Delta,i+1}} C_i(\Delta) \xrightarrow{\boldsymbol{\partial}_{\Delta,i}} C_{i-1}(\Delta) \to \cdots$$

$$\cdots \leftarrow C_{i+1}(\Delta) \xleftarrow{\boldsymbol{\partial}^*_{\Delta,i+1}} C_i(\Delta) \xleftarrow{\boldsymbol{\partial}^*_{\Delta,i}} C_{i-1}(\Delta) \leftarrow \cdots$$

Proof. Since the matrices that represent the maps $\boldsymbol{\partial}$ and $\boldsymbol{\partial}^*$ are mutual transposes, it suffices to prove the first assertion. For $F \in \Delta_i$, we have by (27)

$$\begin{aligned}
\uparrow^a \boldsymbol{\partial}_{\Delta,i-1}(\uparrow^a \boldsymbol{\partial}_{\Delta,i}([F])) \; &= \uparrow^a \boldsymbol{\partial}_{\Delta,i-1} \left(\sum_{j \in F} \varepsilon(j,F) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+1}(x_{F\setminus j})} [F\setminus j] \right) \\
&= \sum_{j \in F} \varepsilon(j,F) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+1}(x_{F\setminus j})} \cdot \uparrow^a \boldsymbol{\partial}_{\Delta,i-1}([F\setminus j]) \\
&= \sum_{j \in F} \varepsilon(j,F) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+1}(x_{F\setminus j})} \sum_{k \in F\setminus j} \varepsilon(k,F\setminus j) \frac{\uparrow^{d-i+a+1}(x_{F\setminus j})}{\uparrow^{d-i+a+2}(x_{F\setminus j\setminus k})} [F\setminus j\setminus k] \\
&= \sum_{j \in F} \sum_{k \in F\setminus j} \varepsilon(j,F)\varepsilon(k,F\setminus j) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+2}(x_{F\setminus j\setminus k})} [F\setminus j\setminus k] \\
&= \sum_{\substack{j,k \in F \\ j \neq k}} \Big(\varepsilon(j,F)\varepsilon(k,F\setminus j) + \varepsilon(k,F)\varepsilon(j,F\setminus k) \Big) \frac{\uparrow^{d-i+a}(x_F)}{\uparrow^{d-i+a+2}(x_{F\setminus j\setminus k})} [F\setminus j\setminus k]
\end{aligned}$$

and it is a standard fact of simplicial homology theory that the parenthesized expression is zero. □

Define the algebraic finely weighted up-down, down-up, and total Laplacians by

$$\mathbf{L}^{\mathrm{ud}}_{\Delta,i} = \boldsymbol{\partial}_{\Delta,i+1} \boldsymbol{\partial}^*_{\Delta,i+1}, \qquad \mathbf{L}^{\mathrm{du}}_{\Delta,i} = \boldsymbol{\partial}^*_{\Delta,i} \boldsymbol{\partial}_{\Delta,i}, \qquad \mathbf{L}^{\mathrm{tot}}_{\Delta,i} = \mathbf{L}^{\mathrm{ud}}_{\Delta,i} + \mathbf{L}^{\mathrm{du}}_{\Delta,i}.$$

Each of these is a linear endomorphism of $C_i(\Delta)$, represented by a symmetric matrix, hence diagonalizable. Let $\mathbf{s}^{ud}_i(\Delta)$, $\mathbf{s}^{du}_i(\Delta)$, and $\mathbf{s}^{tot}_i(\Delta)$ denote the spectra (multisets of eigenvalues) of $\mathbf{L}^{\mathrm{ud}}_{\Delta,i}$, $\mathbf{L}^{\mathrm{du}}_{\Delta,i}$, and $\mathbf{L}^{\mathrm{tot}}_{\Delta,i}$ respectively. We will use the abbreviations $\mathbf{L}^{\mathrm{ud}}$, $\mathbf{L}^{\mathrm{du}}$, $\mathbf{L}^{\mathrm{tot}}$, $\mathbf{s}^{ud}$, $\mathbf{s}^{du}$, $\mathbf{s}^{tot}$ when no confusion can arise.

If we regard $\mathbf{L}^{\mathrm{ud}}_{\Delta,d-1}$ as a matrix with rows and columns indexed by Δ_{d-1}, then it is not hard to check that its (F,G) entry is

$$(\mathbf{L}^{\mathrm{ud}}_{\Delta,d-1})_{FG} = \begin{cases} \varepsilon(j,H)\ \varepsilon(i,H)\ \dfrac{X_H}{\uparrow x_F \ \uparrow x_G} & \text{if } H = F \cup j = G \cup i \in \Delta, \\[2ex] \displaystyle\sum_{j\,:\,F\cup j \in \Delta} \frac{X_{F\cup j}}{\uparrow X_F} & \text{if } F = G, \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{29}$$

This matrix is almost identical to the matrix $N(\Sigma)$ defined in (23) when $\Delta = \mathrm{del}_p \Sigma$, as we now explain.

Lemma 6.3. *Let Σ^d be a pure shifted complex with initial vertex p, and let $\Lambda = \mathrm{link}_p \Sigma$ and $\Delta = \mathrm{del}_p \Sigma$. Then*

$$\hat{\tau}_d(\Sigma) = \left(\prod_{F \in \Lambda_{d-1}} \uparrow X_F \right) \prod_{\lambda \in \mathbf{s}^{ud}_{\Delta,d-1}} (X_{1,p} + \lambda).$$

Proof. First, note that $\mathbf{L} = \mathbf{L}^{\mathrm{ud}}_{\Delta,d-1}$ is indexed by the faces of Δ_{d-1} and $N = N(\Sigma)$ is indexed by the faces of Λ_{d-1}. These indexing sets coincide because Σ is shifted and pure of dimension d.

Second, we show that the off-diagonal entries (the first cases in (23) and (29)) coincide. Suppose that F, G are distinct faces in $\Delta_{d-1} = \Lambda_{d-1}$, and that $H = F \cup i = G \cup j$. Suppose that $i \neq j$ and $H = F \cup i = G \cup j$. We must show that $H \in \Delta$ if and only if $H \in \Sigma$. The "only if" direction is

immediate because $\Delta \subset \Sigma$. On the other hand, $H = F \cup G$ and no element of Δ_{d-1} contains p. Therefore, if $H \in \Sigma$, then $H \in \Delta$, as desired.

Third, we compare the entries on the main diagonals of $\mathbf{L}$ and N. Their only difference is that the summand with $j = p$ occurs in the second case of (23), but not in (29). Hence

$$N_{FF} \;=\; \mathbf{L}_{FF} + \frac{X_{F\cup p}}{\uparrow X_F} \;=\; \mathbf{L}_{FF} + X_{1,p} \tag{30}$$

by (19b). Therefore $N = \mathbf{L} + X_{1,p}I$, where I is an identity matrix of size $f_{d-1}(\Delta)$, and

$$\det N = \chi(-\mathbf{L}, X_{1,p}) = \prod_{\lambda \in \mathbf{s}^{ud}_{\Delta,d-1}} (X_{1,p} + \lambda),$$

where χ denotes the characteristic polynomial of $-\mathbf{L}$ in the variable $X_{1,p}$. The lemma now follows from equation (24). □

The goal of the next two sections is to compute $\mathbf{L}^{\mathrm{ud}}_{\Delta,d-1}$.

7. Cones and near-cones

A shifted complex is an iterated near-cone, so we want to describe the Laplacian eigenvalues of a near-cone in terms of its base. Before we do so, we must consider the case of a cone. Proposition 7.2 provides the desired recurrence for cones, and Proposition 7.6 for near-cones.

7.1. **Boundary and coboundary operators of cones.** Let Γ be the simplicial complex with the single vertex 1, and let Δ^d be any complex on $V = [2, n]$. For a face $F \in \Delta$, write $\tilde{F} = 1 \cup F$. The corresponding *cone* is

$$\Sigma = 1 * \Delta = \Gamma * \Delta = \{F, \tilde{F} \colon F \in \Delta\}.$$

We will make use of the identification

$$C_i(\Sigma) \;\cong\; (C_{-1}(\Gamma) \otimes C_i(\Delta)) \oplus (C_0(\Gamma) \otimes C_{i-1}(\Delta)).$$

By (27) and (28), the (raised) boundary and coboundary maps on Γ are given explicitly by

$$\begin{aligned} \uparrow^a \boldsymbol{\partial}_\Gamma[1] &= x_{a+1,1}[\emptyset], & \uparrow^a \boldsymbol{\partial}^*_\Gamma[1] &= 0, \\ \uparrow^a \boldsymbol{\partial}_\Gamma[\emptyset] &= 0, & \uparrow^a \boldsymbol{\partial}^*_\Gamma[\emptyset] &= x_{a+1,1}[1]. \end{aligned}$$

Next, we give explicit formulas for the maps $\boldsymbol{\partial}_{\Sigma,i}$ and $\boldsymbol{\partial}^*_{\Sigma,i+1}$. How these maps act on a face of Σ depends on whether it is of the form F, for $F \in \Delta_i$, or $\tilde{F}$, for $F \in \Delta_{i-1}$. Note that in any case $\varepsilon(1, \tilde{F}) = 1$, and that for all $v \in F$ we have $\varepsilon(v, F) = -\varepsilon(v, \tilde{F})$. Therefore,

$$\begin{aligned} \boldsymbol{\partial}_{\Sigma,i}([\emptyset] \otimes [F]) &= \sum_{j \in F} \varepsilon(j, F) \frac{\uparrow^{d-i+1}(x_F)}{\uparrow^{d-i+2}(x_{F\backslash j})} [\emptyset] \otimes [F\backslash j] \\ &= \uparrow \left(\sum_{j \in F} \varepsilon(j, F) \frac{\uparrow^{d-i}(x_F)}{\uparrow^{d-i+1}(x_{F\backslash j})} [\emptyset] \otimes [F\backslash j] \right) \\ &= (\mathrm{id} \otimes \uparrow \boldsymbol{\partial}_{\Delta,i})([\emptyset] \otimes [F]), \end{aligned} \tag{31a}$$

$$\begin{aligned}\boldsymbol{\partial}_{\Sigma,i}([1]\otimes[F]) &= \varepsilon(1,\tilde{F})\frac{\uparrow^{d-i+1}(x_{\tilde{F}})}{\uparrow^{d-i+2}(x_F)}[\emptyset]\otimes[F] + \sum_{j\in F}\varepsilon(j,\tilde{F})\frac{\uparrow^{d-i+1}(x_{\tilde{F}})}{\uparrow^{d-i+2}(x_{\tilde{F}\setminus j})}[1]\otimes[F\backslash j]\\ &= x_{d-i+2,1}[\emptyset]\otimes[F] - \sum_{j\in F}\varepsilon(j,F)\frac{x_{d-i+2,1}}{x_{d-i+3,1}}\frac{\uparrow^{d-i+2}(x_F)}{\uparrow^{d-i+3}(x_{F\setminus j})}[1]\otimes[F\backslash j]\\ &= x_{d-i+2,1}[\emptyset]\otimes[F] - \frac{x_{d-i+2,1}}{x_{d-i+3,1}}\uparrow\left(\sum_{j\in F}\varepsilon(j,F)\frac{\uparrow^{d-i+1}(x_F)}{\uparrow^{d-i+2}(x_{F\setminus j})}[1]\otimes[F\backslash j]\right)\\ &= \left(\uparrow^{d-i+1}\boldsymbol{\partial}_\Gamma\otimes\mathrm{id} - \frac{x_{d-i+2,1}}{x_{d-i+3,1}}\,\mathrm{id}\otimes\uparrow\boldsymbol{\partial}_{\Delta,i-1}\right)([1]\otimes[F]),\end{aligned} \tag{31b}$$

$$\begin{aligned}\boldsymbol{\partial}^*_{\Sigma,i+1}([\emptyset]\otimes[F]) &= \varepsilon(1,\tilde{F})\frac{\uparrow^{d-i}(x_{\tilde{F}})}{\uparrow^{d-i+1}(x_F)}[1]\otimes[F] + \sum_{j\in V\setminus F}\varepsilon(j,F\cup j)\frac{\uparrow^{d-i}(x_{F\cup j})}{\uparrow^{d-i+1}(x_F)}[\emptyset]\otimes[F\cup j]\\ &= x_{d-i+1,1}[1]\otimes[F] + \uparrow\left(\sum_{j\in V\setminus F}\varepsilon(j,F\cup j)\frac{\uparrow^{d-i-1}(x_{F\cup j})}{\uparrow^{d-i}(x_F)}[\emptyset]\otimes[F\cup j]\right)\\ &= \Big(\uparrow^{d-i}\boldsymbol{\partial}^*_\Gamma\otimes\mathrm{id} + \mathrm{id}\otimes\uparrow\boldsymbol{\partial}^*_{\Delta,i+1}\Big)([\emptyset]\otimes[F]),\end{aligned} \tag{31c}$$

$$\begin{aligned}\boldsymbol{\partial}^*_{\Sigma,i+1}([1]\otimes[F]) &= \sum_{j\in V\setminus F}\varepsilon(j,\tilde{F}\cup j)\frac{\uparrow^{d-i}(x_{\tilde{F}\cup j})}{\uparrow^{d-i+1}(x_{\tilde{F}\cup j})}[1]\otimes[F\cup j]\\ &= -\sum_{j\in V\setminus F}\varepsilon(j,F\cup j)\frac{\uparrow^{d-i}(x_{1,1})}{\uparrow^{d-i+1}(x_{1,1})}\frac{\uparrow^{d-i+1}(x_{F\cup j})}{\uparrow^{d-i+2}(x_F)}[1]\otimes[F\cup j]\\ &= -\frac{x_{d-i+1,1}}{x_{d-i+2,1}}\cdot\uparrow\left(\sum_{j\in V\setminus F}\varepsilon(j,F\cup j)\frac{\uparrow^{d-i}(x_{F\cup j})}{\uparrow^{d-i+1}(x_F)}[1]\otimes[F\cup j]\right)\\ &= \left(-\frac{x_{d-i+1,1}}{x_{d-i+2,1}}\,\mathrm{id}\otimes\uparrow\boldsymbol{\partial}^*_{\Delta,i}\right)([1]\otimes[F]).\end{aligned} \tag{31d}$$

7.2. **Eigenvectors of cones.** In order to describe the Laplacian eigenvalues and eigenvectors of $1*\Delta$ in terms of those of Δ, we first need some basic facts about the Laplacians of an arbitrary simplicial complex. The following proposition does not depend on fine weighting, and works with any weighted boundary map that satisfies $\partial^2=0$.

Proposition 7.1. *Let Ω^d be a simplicial complex, and let $-1\le i\le d$. Let $\mathbf{L}_i^{\mathrm{ud}}=\mathbf{L}_{\Omega,i}^{\mathrm{ud}}$, $\mathbf{L}_i^{\mathrm{du}}=\mathbf{L}_{\Omega,i}^{\mathrm{du}}$, $\boldsymbol{\partial}_i=\boldsymbol{\partial}_{\Omega,i}$, and $\boldsymbol{\partial}_i^*=\boldsymbol{\partial}_{\Omega,i}^*$.*

(1) *The chain group $C_i(\Omega)$ decomposes as a direct sum*

$$C_i(\Omega)=C_i^{\mathrm{ud}}(\Omega)\oplus C_i^{\mathrm{du}}(\Omega)\oplus C_i^0(\Omega) \tag{32}$$

where

- *$C_i^{\mathrm{ud}}(\Omega)$ has a basis consisting of eigenvectors of $\mathbf{L}_i^{\mathrm{ud}}$ whose eigenvalues are all nonzero, and on which $\mathbf{L}_i^{\mathrm{du}}$ acts by zero;*
- *$C_i^{\mathrm{du}}(\Omega)$ has a basis consisting of eigenvectors of $\mathbf{L}_i^{\mathrm{du}}$ whose eigenvalues are all nonzero, and on which $\mathbf{L}_i^{\mathrm{ud}}$ acts by zero; and*
- *$\mathbf{L}_i^{\mathrm{ud}}$ and $\mathbf{L}_i^{\mathrm{du}}$ both act by zero on $C_i^0(\Omega)$.*

(2) *$\ker(\mathbf{L}_i^{\mathrm{ud}})=\ker(\boldsymbol{\partial}_{i+1}^*)$ and $\ker(\mathbf{L}_i^{\mathrm{du}})=\ker(\boldsymbol{\partial}_i)$.*

(3) *$\dim C_i^0(\Omega)=\tilde{\beta}_i(\Omega)$, the i^{th} Betti number of Ω.*

(4) *Each of the spectra* $\mathbf{s}_i^{ud}$, $\mathbf{s}_i^{du}$, $\mathbf{s}_i^{tot}$ *of* Ω *has cardinality* $f_i(\Omega)$ *as a multiset, and*

$$\mathbf{s}_i^{du} \stackrel{\circ}{=} \mathbf{s}_{i-1}^{ud}, \quad \textit{and} \tag{33a}$$

$$\mathbf{s}_i^{tot} \stackrel{\circ}{=} \mathbf{s}_i^{ud} \cup \mathbf{s}_i^{du} \stackrel{\circ}{=} \mathbf{s}_i^{ud} \cup \mathbf{s}_{i-1}^{ud}. \tag{33b}$$

Proof. For assertion (1), note that $\mathbf{L}_i^{\mathrm{ud}}\mathbf{L}_i^{\mathrm{du}} = \boldsymbol{\partial}\boldsymbol{\partial}^*\boldsymbol{\partial}^*\boldsymbol{\partial} = 0$ and $\mathbf{L}_i^{\mathrm{du}}\mathbf{L}_i^{\mathrm{ud}} = \boldsymbol{\partial}^*\boldsymbol{\partial}\boldsymbol{\partial}\boldsymbol{\partial}^* = 0$. Thus we may simply take $C_i^{\mathrm{ud}}(\Omega)$ and $C_i^{\mathrm{du}}(\Omega)$ to be the spans of the eigenvectors of $\mathbf{L}_i^{\mathrm{ud}}$ and $\mathbf{L}_i^{\mathrm{du}}$ with nonzero eigenvalues.

For assertion (2), the operators $\mathbf{L}_i^{\mathrm{ud}}$ and $\boldsymbol{\partial}_{i+1}^*$ act on the same space, namely $C_i(\Omega)$, and they have the same rank (this is just the linear algebra fact that $\operatorname{rank}(MM^T) = \operatorname{rank} M$ for any matrix M). Therefore, their kernels have the same dimension. Clearly $\ker(\mathbf{L}_i^{\mathrm{ud}}) \supseteq \ker(\boldsymbol{\partial}_{i+1}^*)$, so we must have equality. The same argument shows that $\ker(\mathbf{L}_i^{\mathrm{du}}) = \ker(\boldsymbol{\partial}_i)$.

Assertion (3) follows from the calculation

$$\begin{aligned}\tilde{\beta}_i(\Omega) \;=\; \dim \tilde{H}_i(\Omega,\mathbb{Q}) &= (\dim\ker \boldsymbol{\partial}_i) - (\dim \operatorname{im} \boldsymbol{\partial}_{i+1}) \\ &= (f_i - \operatorname{rank}\boldsymbol{\partial}_i) - (\operatorname{rank}\boldsymbol{\partial}_{i+1}) \\ &= \dim C_i - \dim C_i^{\mathrm{ud}} - \dim C_i^{\mathrm{du}} \;=\; \dim C_i^0.\end{aligned}$$

For assertion (4), first note that $\dim C_i(\Omega) = f_i(\Omega)$, and that the matrix representing each spectrum is symmetric, hence diagonalizable. The identity (33a) is a standard fact in linear algebra, and (33b) is a consequence of the decomposition (32). □

Proposition 7.2. *As in Section 7.1, let* Δ^d *be a simplicial complex on vertex set* $[2,n]$*, and let* $\Sigma = 1 * \Delta$*. Then*

$$\begin{aligned}\mathbf{s}_i^{ud}(\Sigma) \stackrel{\circ}{=} &\left\{X_{d-i+1,1} + \uparrow\lambda \colon \lambda \in \mathbf{s}_i^{ud}(\Delta),\ \lambda \neq 0\right\} \\ &\cup \left\{X_{d-i+1,1} + \frac{X_{d-i+1,1}}{X_{d-i+2,1}} \uparrow\mu \colon \mu \in \mathbf{s}_{i-1}^{ud}(\Delta),\ \mu \neq 0\right\} \\ &\cup \left\{X_{d-i+1,1}\right\}^{\tilde{\beta}_i(\Delta)}.\end{aligned} \tag{34}$$

Here the symbol $\cup$ denotes multiset union, and the superscript in the last line indicates multiplicity.

Proof. Throughout the proof, we abbreviate $\mathbf{L}_{\Sigma,i}^{\mathrm{ud}}$ by $\mathbf{L}$. All other Laplacians that arise will be specified precisely.

First, let $V \in C_i^{\mathrm{ud}}(\Delta)$ be an eigenvector of $\mathbf{L}_{\Delta,i}^{\mathrm{ud}}$ with eigenvalue $\lambda \neq 0$. Then $\uparrow\mathbf{L}_{\Delta,i}^{\mathrm{ud}}(\uparrow V) = \uparrow\lambda \uparrow V$, and, by Lemma 6.2,

$$\uparrow\boldsymbol{\partial}_{\Delta,i}(\uparrow V) = \frac{1}{\uparrow\lambda} \uparrow\boldsymbol{\partial}_{\Delta,i}\big(\uparrow\boldsymbol{\partial}_{\Delta,i+1}(\uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow V))\big) = 0. \tag{35}$$

Using (31a)…(31d) and (35), we calculate

$$\begin{aligned}\mathbf{L}\left([\emptyset]\otimes\uparrow V\right) &= \boldsymbol{\partial}_{\Sigma,i}(\boldsymbol{\partial}_{\Sigma,i}^*([\emptyset]\otimes\uparrow V)) \\ &= \boldsymbol{\partial}_{\Sigma,i}\left(\uparrow^{d-i}\boldsymbol{\partial}_\Gamma^*[\emptyset]\otimes\uparrow V + [\emptyset]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow V)\right) \\ &= x_{d-i+1,1}\boldsymbol{\partial}_{\Sigma,i+1}\left([1]\otimes\uparrow V\right) + \boldsymbol{\partial}_{\Sigma,i+1}\left([\emptyset]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow V)\right) \\ &= x_{d-i+1,1}\left(\uparrow^{d-i}\boldsymbol{\partial}_\Gamma[1]\otimes\uparrow V - \frac{x_{d-i+1,1}}{x_{d-i+2,1}}[1]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i}(\uparrow V)\right) \\ &\quad + [\emptyset]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i+1}(\uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow V)) \\ &= \left(X_{d-i+1,1} + \uparrow\lambda\right)\left([\emptyset]\otimes\uparrow V\right).\end{aligned} \tag{36}$$

Therefore, $[\emptyset]\otimes\uparrow V \in C_i(\Sigma)$ is an eigenvector of $\mathbf{L}$, with eigenvalue $X_{d-i+1,1} + \uparrow\lambda$. Notice that this eigenvalue cannot be zero: since $1 \notin \Delta$, no Laplacian eigenvalue of Δ can possibly equal $-X_{d-i,1}$.

Second, let W be an eigenvector in $C_i^{\mathrm{du}}(\Delta)$ with nonzero eigenvalue μ. That is, $\mathbf{L}_{\Delta_i}^{\mathrm{du}}(W) = \boldsymbol{\partial}_{\Delta,i}^*(\boldsymbol{\partial}_{\Delta,i}(W)) = \mu W$, and $\boldsymbol{\partial}_{\Delta,i+1}^*(W) = 0$ by a computation similar to (35). Define

$$A = [\emptyset]\otimes \uparrow W, \qquad B = [1]\otimes \uparrow \boldsymbol{\partial}_{\Delta,i}(\uparrow W).$$

Note that these are both nonzero elements of $C_i(\Sigma)$. Then

$$\begin{aligned}
\mathbf{L}(A) &= \boldsymbol{\partial}_{\Sigma,i+1}\left(\boldsymbol{\partial}_{\Sigma,i+1}^*([\emptyset]\otimes \uparrow W)\right)\\
&= \boldsymbol{\partial}_{\Sigma,i+1}\left(\uparrow^{d-i}\boldsymbol{\partial}_{\Gamma}^*[\emptyset]\otimes \uparrow W + [\emptyset]\otimes \uparrow \boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow W)\right)\\
&= x_{d-i+1,1}\boldsymbol{\partial}_{\Sigma,i+1}\left([1]\otimes \uparrow W\right) \qquad (37a)\\
&= x_{d-i+1,1}\left(\uparrow^{d-i}\boldsymbol{\partial}_{\Gamma}[1]\otimes \uparrow W - \frac{x_{d-i+1,1}}{x_{d-i+2,1}}[1]\otimes \uparrow \boldsymbol{\partial}_{\Delta,i}(\uparrow W)\right)\\
&= X_{d-i+1,1}A - \left(\frac{X_{d-i+1,1}}{x_{d-i+2,1}}\right)B \qquad (37b)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{L}(B) &= \boldsymbol{\partial}_{\Sigma,i+1}\left(\boldsymbol{\partial}_{\Sigma,i+1}^*([1]\otimes \uparrow \boldsymbol{\partial}_{\Delta,i}(\uparrow W))\right)\\
&= \boldsymbol{\partial}_{\Sigma,i+1}\left(-\frac{x_{d-i+1,1}}{x_{d-i+2,1}}[1]\otimes \uparrow \boldsymbol{\partial}_{\Delta,i}^*(\uparrow \boldsymbol{\partial}_{\Delta,i}(\uparrow W))\right)\\
&= -\frac{x_{d-i+1,1}}{x_{d-i+2,1}}\uparrow\mu \cdot \boldsymbol{\partial}_{\Sigma,i+1}\left([1]\otimes \uparrow W\right)\\
&= \left(-\frac{X_{d-i+1,1}}{x_{d-i+2,1}}\uparrow\mu\right)A + \left(\frac{X_{d-i+1,1}}{X_{d-i+2,1}}\uparrow\mu\right)B,
\end{aligned}$$

where the last step follows by the equality of (37a) and (37b). Letting

$$f = X_{d-i+1,1}, \qquad g = -\frac{X_{d-i+1,1}}{x_{d-i+2,1}}, \qquad h = -\frac{\uparrow\mu}{x_{d-i+2,1}},$$

the calculations above say that

$$\mathbf{L}(A) = fA + gB, \qquad \mathbf{L}(B) = h(fA+gB),$$

which implies that

$$\mathbf{L}(fA+gB) = f\mathbf{L}(A) + g\mathbf{L}(B) = f(fA+gB) + gh(fA+gB) = (f+gh)(fA+gB).$$

That is, $fA+gB$ is an eigenvector of $\mathbf{L}$ with eigenvalue

$$f + gh = X_{d-i+1,1} + \frac{X_{d-i+1,1}}{X_{d-i+2,1}}\uparrow\mu.$$

As before, this quantity cannot be zero because Δ does not contain the vertex 1.

Third, let $Z \in C_i^0(\Delta)$; we will show that $[\emptyset]\otimes\uparrow Z$ is an eigenvector of $\mathbf{L}$ with eigenvalue $X_{d-i+1,1}$. Indeed, by (2) of Proposition 7.1, we have $\boldsymbol{\partial}_{\Delta,i}(Z) = \boldsymbol{\partial}_{\Delta,i+1}^*(Z) = 0$, so that

$$\uparrow\boldsymbol{\partial}_{\Delta,i}(\uparrow Z) = \uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow Z) = 0.$$

Therefore

$$\begin{aligned}
\mathbf{L}([\emptyset]\otimes\uparrow Z) &= \boldsymbol{\partial}_{\Sigma,i+1}(\boldsymbol{\partial}_{\Sigma,i+1}^*([\emptyset]\otimes\uparrow Z))\\
&= \boldsymbol{\partial}_{\Sigma,i+1}\left((\uparrow^{d-i}\boldsymbol{\partial}_{\Gamma}^*[\emptyset]\otimes\uparrow Z) + [\emptyset]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i+1}^*(\uparrow Z)\right)\\
&= x_{d-i+1,1}\boldsymbol{\partial}_{\Sigma,i+1}([1]\otimes\uparrow Z)\\
&= x_{d-i+1,1}\cdot\left(\uparrow^{d-i}\boldsymbol{\partial}_{\Gamma}[1]\otimes\uparrow Z - \frac{x_{d-i+1,1}}{x_{d-i+2}}[1]\otimes\uparrow\boldsymbol{\partial}_{\Delta,i}(\uparrow Z)\right)\\
&= X_{d-i+1,1}([\emptyset]\otimes\uparrow Z), \qquad (38)
\end{aligned}$$

as desired. By assertion (3) of Proposition 7.1, the multiplicity of this eigenvalue is $\dim C_i^0(\Delta) = \tilde{\beta}_i(\Delta)$.

At this point, we have proven that (34) is true if "$\overset{\circ}{=}$" is replaced with "$\supseteq$". On the other hand, we have accounted for

$$\dim C_i^{\mathrm{ud}}(\Delta) + \dim C_i^{\mathrm{du}}(\Delta) + \dim C_i^0(\Delta) = \dim C_i(\Delta) = f_i(\Delta)$$

nonzero eigenvalues in $\mathbf{s}_i^{ud}(\Sigma)$. Since the preceding calculations hold for all i, we also know $f_{i-1}(\Delta)$ nonzero eigenvalues of $\mathbf{L}_{\Sigma,i-1}^{\mathrm{ud}}$ ($\overset{\circ}{=} \mathbf{L}_{\Sigma,i}^{\mathrm{du}}$). By (33b), we have accounted for all $f_i(\Delta) + f_{i-1}(\Delta) = f_i(\Sigma) = \dim C_i(\Sigma)$ eigenvalues of $\mathbf{L}_{\Sigma,i}^{\mathrm{tot}}$. So we have indeed found all the nonzero eigenvalues of $\mathbf{L}$. □

One can obtain explicit formulas for the spectra $\mathbf{s}_i^{du}(\Sigma)$ and $\mathbf{s}_i^{tot}(\Sigma)$ by applying (33a) and (33b) to the formula (34); we omit the details.

7.3. **Eigenvalues of near-cones.** The next step is to establish a recurrence (Proposition 7.6) for the Laplacian eigenvalues of near-cones, in terms of the eigenvalues of the deletion and the link of the apex. Our method is based on that of Lemma 5.3 of [13]. By itself, Proposition 7.6 is not a proper recurrence, in the sense that it computes the eigenvalues of a pure complex in terms of complexes that are not necessarily pure. Therefore, it cannot be applied recursively; the proper recurrence for shifted complexes will have to wait until Theorem 8.2. Since we will be comparing complexes with similar face sets but of different dimensions, we begin by describing how their Laplacian spectra are related.

Lemma 7.3. *Let Σ^d be a simplicial complex, and let $j < i \leq d$. Then $\mathbf{s}_j^{ud}(\Sigma) = \uparrow^{d-i} \mathbf{s}_j^{ud}(\Sigma_{(i)})$.*

Proof. The complexes $\Sigma_{(i)}$ and Σ have the same face sets for every dimension $\leq i$, but $\dim \Sigma_{(i)} = \dim \Sigma - (d-i)$. Therefore, the algebraic finely weighted boundary maps and Laplacians of Σ can be obtained from those of $\Sigma_{(i)}$ by applying $\uparrow^{d-i}$, from which the lemma follows. □

Recall that the *pure i-skeleton* of Σ is the subcomplex $\Sigma_{[i]}$ generated by the i-dimensional faces of Σ.

Lemma 7.4. *Let Σ^d be a simplicial complex. Then $\mathbf{s}_{d-1}^{ud}(\Sigma) \overset{\circ}{=} \mathbf{s}_{d-1}^{ud}(\Sigma_{[d]})$.*

Proof. This result is proved in [12, Lemma 3.2], but we sketch the proof here for completeness. First, observe that $\mathbf{L}_{d-1}^{\mathrm{ud}}$ depends only on $(d-1)$- and d-dimensional faces. Letting $\Omega = \Sigma_{[d]}$, we have $\Omega_d = \Sigma_d$ and $\Omega_{d-1} \subseteq \Sigma_{d-1}$, and indeed $\mathbf{L}_{\Sigma,d-1}^{\mathrm{ud}}[F] = \mathbf{L}_{\Omega,d-1}^{\mathrm{ud}}[F]$ for any $F \in \Omega_{d-1}$. On the other hand, $\Sigma_{d-1} \backslash \Omega_{d-1}$ consists precisely of those faces G not contained in any d-dimensional faces of Σ. But $\mathbf{L}_{\Sigma,d-1}^{\mathrm{ud}}$ acts by zero on any such G, and the lemma follows immediately. □

Corollary 7.5. *Let Σ^d be a simplicial complex. Then $\mathbf{s}_{i-1}^{ud}(\Sigma) \overset{\circ}{=} \uparrow^{d-i} \mathbf{s}_{i-1}^{ud}(\Sigma_{[i]})$.*

Proof. We have $\mathbf{s}_{i-1}^{ud}(\Sigma) = \uparrow^{d-i} \mathbf{s}_{i-1}^{ud}(\Sigma_{(i)}) \overset{\circ}{=} \uparrow^{d-i} \mathbf{s}_{i-1}^{ud}((\Sigma_{(i)})_{[i]}) = \uparrow^{d-i} \mathbf{s}_{i-1}^{ud}(\Sigma_{[i]})$ by Lemmas 7.3 and 7.4. (Note that $(\Sigma_{(i)})_{[i]} = \Sigma_{[i]}$.) □

Proposition 7.6. *Let Σ^d be a pure near-cone with apex p, and let $\Delta = \mathrm{del}_p \Sigma$ and $\Lambda = \mathrm{link}_p \Sigma$ be the deletion and link, respectively, of Σ with respect to vertex p. Then*

$$\begin{aligned}\mathbf{s}_{d-1}^{ud}(\Sigma) \overset{\circ}{=} &\left\{X_{1,p} + \lambda \colon \lambda \in \mathbf{s}_{d-1}^{ud}(\Delta),\ \lambda \neq 0\right\} \\ &\cup \left\{X_{1,p} + \frac{X_{1,p}}{X_{2,p}} \uparrow\mu \colon \mu \in \mathbf{s}_{d-2}^{ud}(\Lambda),\ \mu \neq 0\right\} \\ &\cup \left\{X_{1,p}\right\}^{\tilde{\beta}_{d-1}(\Delta)}.\end{aligned}$$

Proof. If Σ is a cone, then the result follows from direct application of Proposition 7.2. (Note that $\mathbf{s}^{ud}_{d-1}(\Delta)$ consists of only 0's in this case, since Δ is only $(d-1)$-dimensional.) Thus, we may as well assume for the remainder of the proof that Σ is not a cone. In this case, $\dim \Delta = d$ and $\dim \Lambda = d-1$. It is not difficult to see that

$$\Delta_{(d-1)} = \Lambda, \text{ and} \tag{39}$$
$$\Sigma_{(d)} = \Sigma = (p * \Delta)_{(d)}. \tag{40}$$

Applying Lemma 7.3 to equation (40) (keeping in mind that $\dim(p*\Delta) = d+1$), and then applying Proposition 7.2, we find that

$$\begin{aligned}
\uparrow \mathbf{s}^{ud}_{d-1}(\Sigma) &= \mathbf{s}^{ud}_{d-1}(p*\Delta) \\
&\stackrel{\circ}{=} \left\{X_{2,p} + \uparrow\lambda \colon \lambda \in \mathbf{s}^{ud}_{d-1}(\Delta),\ \lambda \neq 0\right\} \\
&\quad \cup \left\{X_{2,p} + \frac{X_{2,p}}{X_{3,p}} \uparrow\mu \colon \mu \in \mathbf{s}^{ud}_{d-2}(\Delta),\ \mu \neq 0\right\} \\
&\quad \cup \{X_{2,p}\}^{\tilde{\beta}_{d-1}(\Delta)}
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{s}^{ud}_{d-1}(\Sigma) &= \left\{X_{1,p} + \lambda \colon \lambda \in \mathbf{s}^{ud}_{d-1}(\Delta),\ \lambda \neq 0\right\} \\
&\quad \cup \left\{X_{1,p} + \frac{X_{1,p}}{X_{2,p}}\mu \colon \mu \in \mathbf{s}^{ud}_{d-2}(\Delta),\ \mu \neq 0\right\} \\
&\quad \cup \{X_{1,p}\}^{\tilde{\beta}_{d-1}(\Delta)}.
\end{aligned}$$

The desired result now follows from applying Lemma 7.3 to equation (39). □

8. The Laplacian spectrum of a shifted complex

In this section, we explicitly describe the eigenvalues of the algebraic finely weighted Laplacians of a shifted complex Σ. The eigenvalues are Laurent polynomials called *z-polynomials*, which are in one-to-one correspondence with the *critical pairs* of the complex: pairs (A, B) such that $A \in \Sigma$, $B \notin \Sigma$, and B covers A in the componentwise partial order. The main result, Theorem 1.5, is proved by establishing identical recurrences for the z-polynomials (Theorem 8.2) and critical pairs (Corollary 8.8).

8.1. **z-polynomials.** Let S and T be multisets of integers. Define a Laurent polynomial $z(S,T)$ by the formula

$$z(S,T) = \frac{1}{\uparrow X_S} \sum_{j \in T} X_{S \cup j}, \tag{41}$$

where as usual the symbol $\cup$ denotes multiset union. An example was given at the end of Example 1.7.

Proposition 8.1. *Let $d > i$ be integers, and let S, T be sets of integers greater than p. Then*

$$X_{d-i+1,p} + \uparrow^{d-i} z(S,T) \ = \uparrow^{d-i} z(S, \tilde{T}) \tag{42}$$

and

$$X_{d-i+1,p} + \frac{X_{d-i+1,p}}{X_{d-i+2,p}} \uparrow^{d-i+1} z(S,T) \ = \uparrow^{d-i} z(\tilde{S}, \tilde{T}) \tag{43}$$

where $\tilde{S} = S \cup \{p\}$ and $\tilde{T} = T \cup \{p\}$.

Proof. We will use the identity (19a) repeatedly in the calculations. For (42), observe that

$$\begin{aligned}
X_{d-i+1,p} + \uparrow^{d-i} z(S,T) &= X_{d-i+1,p} + \frac{1}{\uparrow^{d-i+1} X_S} \sum_{j \in T} \uparrow^{d-i} X_{S \cup j} \\
&= \frac{1}{\uparrow^{d-i+1} X_S} \left(X_{d-i+1,p} \cdot \uparrow^{d-i+1} X_S + \sum_{j \in T} \uparrow^{d-i} X_{S \cup j} \right) \\
&= \frac{1}{\uparrow^{d-i+1} X_S} \left(\uparrow^{d-i} X_{\tilde{S}} + \sum_{j \in T} \uparrow^{d-i} X_{S \cup j} \right) \\
&= \frac{1}{\uparrow^{d-i+1} X_S} \sum_{j \in \tilde{T}} \uparrow^{d-i} X_{S \cup j} \\
&= \uparrow^{d-i} z(S, \tilde{T}),
\end{aligned}$$

and for (43), observe that

$$\begin{aligned}
X_{d-i+1,p} + \frac{X_{d-i+1,p}}{X_{d-i+2,p}} \uparrow^{d-i+1} z(S,T) &= X_{d-i+1,p} + \frac{X_{d-i+1,p}}{X_{d-i+2,p} \cdot \uparrow^{d-i+2} X_S} \sum_{j \in T} \uparrow^{d-i+1} X_{S \cup j} \\
&= X_{d-i+1,p} + \frac{X_{d-i+1,p}}{\uparrow^{d-i+1} X_{\tilde{S}}} \sum_{j \in T} \uparrow^{d-i+1} X_{S \cup j} \\
&= \frac{1}{\uparrow^{d-i+1} X_{\tilde{S}}} \left(X_{d-i+1,p} \cdot \uparrow^{d-i+1} X_{\tilde{S}} + \sum_{j \in T} X_{d-i+1,p} \cdot \uparrow^{d-i+1} X_{S \cup j} \right) \\
&= \frac{1}{\uparrow^{d-i+1} X_{\tilde{S}}} \left(\uparrow^{d-i} X_{\tilde{S} \cup p} + \sum_{j \in T} \uparrow^{d-i} X_{\tilde{S} \cup j} \right) \\
&= \frac{1}{\uparrow^{d-i+1} X_{\tilde{S}}} \sum_{j \in \tilde{T}} \uparrow^{d-i} X_{\tilde{S} \cup j} \\
&= \uparrow^{d-i} z(\tilde{S}, \tilde{T}).
\end{aligned}$$

□

Theorem 8.2. *Let Σ^d be a shifted simplicial complex. Then every nonzero eigenvalue of $\mathbf{s}^{ud}_{i-1}(\Sigma)$ has the form of a z-polynomial. Moreover, the spectrum $\mathbf{s}^{ud}_{i-1}(\Sigma)$ is determined recursively as follows.*

If Σ has no vertices, then $\mathbf{s}^{ud}_{i-1}(\Sigma)$ has no nonzero elements.

Otherwise,

$$\begin{aligned}
\mathbf{s}^{ud}_{i-1}(\Sigma) \stackrel{\circ}{=} & \left\{ \uparrow^{d-i} z(S, \tilde{T}) \colon 0 \neq z(S,T) \in \mathbf{s}^{ud}_{i-1}(\Delta) \right\} \\
& \cup \left\{ \uparrow^{d-i} z(\tilde{S}, \tilde{T}) \colon 0 \neq z(S,T) \in \mathbf{s}^{ud}_{i-2}(\Lambda) \right\} \cup \left\{ \uparrow^{d-i} z(\emptyset, \tilde{\emptyset}) \right\}^{\tilde{\beta}_{i-1}(\Delta)}
\end{aligned} \tag{44}$$

where p is the initial vertex of $\Sigma_{[i]}$; $\tilde{S} = S \cup p$; $\tilde{T} = T \cup p$; $\Delta = \operatorname{del}_p \Sigma_{[i]}$; and $\Lambda = \operatorname{link}_p \Sigma_{[i]}$.

Proof. The proof is by induction on the number of vertices of Σ. When Σ has no vertices, Σ is either the empty complex with no faces, or the trivial complex whose only face is the empty face. In either case $\mathbf{s}^{ud}_{i-1}(\Sigma)$ has no nonzero elements.

We now assume that Σ has at least one vertex. Then

$$\begin{aligned}
\mathbf{s}^{ud}_{i-1}(\Sigma_{[i]}) &\stackrel{\circ}{=} \{X_{1,p}+\lambda\colon \lambda\in\mathbf{s}^{ud}_{i-1}(\Delta),\ \lambda\neq 0\}\\
&\quad\cup\left\{X_{1,p}+\frac{X_{1,p}}{X_{2,p}}\uparrow\mu\colon \mu\in\mathbf{s}^{ud}_{i-2}(\Lambda),\ \mu\neq 0\right\}\\
&\quad\cup\{X_{1,p}+0\}^{\tilde\beta_{i-1}(\Delta)}\\
&=\{X_{1,p}+z(S,T)\colon z(S,T)\in\mathbf{s}^{ud}_{i-1}(\Delta),\ z(S,T)\neq 0\}\\
&\quad\cup\left\{X_{1,p}+\frac{X_{1,p}}{X_{2,p}}z(S,T)\colon z(S,T)\in\mathbf{s}^{ud}_{i-2}(\Lambda),\ z(S,T)\neq 0\right\}\\
&\quad\cup\{X_{1,p}+z(\emptyset,\emptyset)\}^{\tilde\beta_{i-1}(\Delta)}\\
&=\left\{z(S,\tilde T)\colon 0\neq z(S,T)\in\mathbf{s}^{ud}_{i-1}(\Delta)\right\}\\
&\quad\cup\left\{z(\tilde S,\tilde T)\colon 0\neq z(S,T)\in\mathbf{s}^{ud}_{i-2}(\Lambda)\right\}\\
&\quad\cup\left\{z(\emptyset,\tilde\emptyset)\right\}^{\tilde\beta_{i-1}(\Delta)}.
\end{aligned}$$

The $\stackrel{\circ}{=}$-equivalence above is by Proposition 7.6. The following equality is justified by the identity $z(\emptyset,\emptyset)=0$ and induction on the number of vertices, since Δ and Λ each have one fewer vertex than Σ. Note that λ and μ are each replaced by $z(S,T)$, with no raising operator, because $\dim\Delta\le i$ and $\dim\Lambda=i-1$. The final equality comes from Proposition 8.1. The result now follows from Corollary 7.5. □

8.2. **Critical pairs.** Throughout this section, let $\mathcal{F}$ be a k-family of sets of integers, and let p be the smallest integer occurring in any element of $\mathcal{F}$.

Definition 8.3. A *critical pair* for $\mathcal{F}$ is an ordered pair (A,B), where $A=\{a_1<a_2<\cdots<a_k\}$ and $B=\{b_1<b_2<\cdots<b_k\}$ are sets of integers such that $A\in\mathcal{F}$, $B\notin\mathcal{F}$, and B covers A in componentwise order. That is, $b_i=a_i+1$ for exactly one i, and $b_j=a_j$ for all $j\neq i$. (Note that b_i need not be in the vertex set of $\mathcal{F}$.) The *signature* of (A,B) is the set of vertices $\sigma(A,B)=\{a_1,\dots,a_{i-1},a_i\}$. The *long signature* is the ordered pair of sets $\bar\sigma(A,B)=(S,T)$, where $S=\{a_1,\dots,a_{i-1}\}$ and $T=\{j\colon p\le j\le a_i\}$. The multisets of signatures and long signatures of critical pairs of $\mathcal{F}$ are denoted $\sigma(\mathcal{F})$ and $\bar\sigma(\mathcal{F})$ respectively.

As described in the Introduction and Example 1.7, critical pairs are especially significant for shifted simplicial complexes. We will soon see that the critical pairs of a shifted complex are in bijection with the eigenvalues of its algebraic finely weighted Laplacian.

Definition 8.4. The *degree* of vertex v in the family $\mathcal{F}$ is $\deg_{\mathcal{F}}(v)=|\{F\in\mathcal{F}\colon v\in F\}|$.

Proposition 8.5. *Let $\mathcal{F}$ be a shifted family. Then $\deg_{\mathcal{F}}(v)-\deg_{\mathcal{F}}(v+1)$ counts the number of signatures of $\mathcal{F}$ whose greatest element is v.*

Proof. Let $S=\{F\in\mathcal{F}\colon v\in F\}$ and $T=\{F\in\mathcal{F}\colon v+1\in F\}$. Partition $S=S_1\,\dot\cup\,S_2\,\dot\cup\,S_3$ and $T=T_1\,\dot\cup\,T_2\,\dot\cup\,T_3$ as follows:

$$\begin{aligned}
S_1&=\{F\in\mathcal{F}\colon v,v+1\in F\},\\
S_2&=\{F\in\mathcal{F}\colon v\in F,\ v+1\notin F,\ F\backslash\{v\}\,\dot\cup\,\{v+1\}\in\mathcal{F}\},\\
S_3&=\{F\in\mathcal{F}\colon v\in F,\ v+1\notin F,\ F\backslash\{v\}\,\dot\cup\,\{v+1\}\notin\mathcal{F}\},\\
\\
T_1&=\{F\in\mathcal{F}\colon v,v+1\in F\},\\
T_2&=\{F\in\mathcal{F}\colon v\notin F,\ v+1\in F,\ F\backslash\{v+1\}\,\dot\cup\,\{v\}\in\mathcal{F}\},\\
T_3&=\{F\in\mathcal{F}\colon v\notin F,\ v+1\in F,\ F\backslash\{v+1\}\,\dot\cup\,\{v\}\notin\mathcal{F}\}.
\end{aligned}$$

Then $S_1 = T_1$, and there is an obvious bijection between S_2 and T_2. Since $\mathcal{F}$ is shifted, $T_3 = \emptyset$, so $\deg_{\mathcal{F}}(v) - \deg_{\mathcal{F}}(v+1) = |S| - |T| = |S_3|$.

Finally, if $F \in S_3$, then (F, G) is a critical pair, where $G = F \backslash \{v\} \dot\cup \{v+1\}$. For each such critical pair, the greatest element of the signature is v. Conversely, if (A, B) is a critical pair whose signature's greatest element is v, then $A \in S_3$. □

Corollary 8.6. *Let Σ^d be a pure shifted simplicial complex with initial vertex p. Then* $\deg_{\Sigma_d}(p) - \deg_{\Sigma_d}(p+1) = \tilde\beta_{d-1}(\operatorname{del}_p \Sigma)$.

Proof. Just as in the proof of Proposition 8.5 above,

$$\deg_{\Sigma_d}(p) - \deg_{\Sigma_d}(p+1) = |S_3|,$$

where

$$S_3 = \{F \in \Sigma_d \colon p \in F,\ p+1 \notin F,\ F \backslash \{p\} \dot\cup \{p+1\} \notin \Sigma_d\}.$$

There is a bijection between S_3 and the set

$$S_3' = \{G \in (\operatorname{del}_p \Sigma)_{d-1} \colon p+1 \notin G,\ G \dot\cup \{p+1\} \notin \operatorname{del}_p \Sigma\}$$

given by $G = F \backslash \{p\}$. Then, by equation (16),

$$\tilde\beta_{d-1}(\operatorname{del}_p \Sigma) = |S_3'|.$$

Combining the four displayed equations yields the desired result. □

Proposition 8.7. *Let Σ be a shifted complex with initial vertex p. Let $\Delta = \operatorname{del}_p \Sigma$ and $\Lambda = \operatorname{link}_p \Sigma$. Then*

$$\sigma(\Sigma_i) = \sigma(\Delta_i) \cup \{p \dot\cup F \colon F \in \sigma(\Lambda_{i-1})\} \cup \{p\}^{\deg_{\Sigma_i}(p) - \deg_{\Sigma_i}(p+1)},$$

where $\cup$ denotes multiset union.

Proof. The multiplicity of the signature $\{p\}$ follows from Proposition 8.5.

Suppose that $A \in \Delta_i$, $B \notin \Delta_i$, and B covers A in the componentwise partial order. Then $A \in \Sigma_i$ and $p \notin A$. Since $p \notin A$ and $A \prec B$, we conclude that $p \notin B$, and so $B \notin \Sigma_i$. Hence we have a one-to-one map of multisets

$$\sigma(\Delta_i) \to \sigma(\Sigma_i).$$

On the other hand, suppose that $A \in \Lambda_{i-1}$, $B \notin \Lambda_{i-1}$, and B covers A in the componentwise partial order. Then $p \notin A$ and $\tilde A = A \dot\cup \{p\} \in \Sigma_i$. The critical pair (A, B) of Λ_{i-1} gives rise to the critical pair $(\tilde A, \tilde B)$ of Σ_i, and it is clear that $\sigma(\tilde A, \tilde B) = \sigma(A, B) \dot\cup \{p\}$. Furthermore, $\tilde B \notin \Sigma_i$ because $B \notin \Lambda_{i-1}$. Hence we have a one-to-one map of multisets

$$\{p \dot\cup F \colon F \in \sigma(\Lambda_{i-1})\} \to \sigma(\Sigma_i).$$

Now we must show, conversely, that every signature $F \neq \{p\}$ of Σ_i arises in one of these two ways. Let F be such a signature of Σ_i, with critical pair (A, B). So $A \in \Sigma_i$; $B \notin \Sigma_i$ (so $B \notin \Delta_i$); and B covers A in the componentwise partial order.

First, if $p \notin F$, then $p \notin A$ and so $A \in \Delta_i$. Thus (A, B) is a critical pair for Δ_i.

Second, if $p \in F$, then $F = F' \dot\cup \{p\}$ for some $F' \neq \emptyset$. In this case, we have $p \in A$ and $p \in B$ for the critical pair (A, B) whose signature is F. Indeed, $p \in F$ directly implies that $p \in A$. If $p \notin B$, then $F = \sigma(A, B) = \{p\}$. Accordingly, let $A' = A \backslash \{p\}$ and $B' = B \backslash \{p\}$. Then $A' \in \Lambda_{i-1}$ and $B \notin \Sigma_i$, so $B' \notin \Lambda_{i-1}$, and (A, B) is a critical pair for Λ_{i-1}. □

Corollary 8.8. *Let Σ^d be a shifted simplicial complex, and let $i \leq d$. Then the multiset $\bar\sigma(\Sigma_i)$ is determined by the following recurrence.*

If Σ has no vertices, then $\bar\sigma(\Sigma_i) = \emptyset$.

Otherwise,

$$\begin{aligned}\bar{\sigma}(\Sigma_i) = \{(S,\tilde{T})\colon (S,T) \in \bar{\sigma}(\Delta_i)\} \\ \cup \{(\tilde{S},\tilde{T})\colon (S,T) \in \bar{\sigma}(\Lambda_{i-1})\} \\ \cup \{(\emptyset,\tilde{\emptyset})\}^{\tilde{\beta}_{i-1}(\Delta)}\end{aligned} \tag{45}$$

where p is the initial vertex of $\Sigma_{[i]}$; $\tilde{S} = S \cup p$; $\tilde{T} = T \cup p$; $\Delta = \operatorname{del}_p \Sigma_{[i]}$; and $\Lambda = \operatorname{link}_p \Sigma_{[i]}$.

Proof. If $\sigma(A,B) = \{p\}$, then $\bar{\sigma}(A,B) = (\emptyset,\{p\}) = (\emptyset,\tilde{\emptyset})$. So the third term in (45) arises from applying Corollary 8.6 to the pure i-dimensional shifted complex $\Sigma_{[i]}$. (Indeed, $\Sigma_{[i]}$ is shifted when Σ is, or even just when Σ_i is.)

For the first two terms in (45), the only hard part is to note that if the first vertex of Σ_i is p, then $p+1$ is the first vertex of Δ_i and Λ_{i-1} (unless $\Delta_i = \emptyset$, but in that case we don't have to worry about the first set). This accounts for the $\tilde{T}$'s.

Even though Proposition 8.7 above defines Δ and Λ somewhat differently than here, it is not a problem because $(\operatorname{del}_p \Sigma)_i = (\operatorname{del}_p \Sigma_{[i]})_i$ and $(\operatorname{link}_p \Sigma)_{i-1} = (\operatorname{link}_p \Sigma_{[i]})_{i-1}$. Then, since Δ and Λ only appear as Δ_i and Λ_{i-1}, it doesn't matter whether we set them to be the deletion and link of Σ or of $\Sigma_{[i]}$. □

8.3. **Theorem 1.5 and its consequences.** We can finally prove Theorem 1.5, which characterizes the Laplacian spectra $\mathbf{s}^{ud}_{i-1}(\Sigma)$ in terms of z-polynomials of critical pairs. In the notation we have developed, the theorem can be restated as follows:

Theorem 1.5. *Let Σ^d be a shifted simplicial complex. Then, for each $0 \le i \le d$,*

$$\mathbf{s}^{ud}_{i-1}(\Sigma) \stackrel{\circ}{=} \uparrow^{d-i}\{z(S,T)\colon (S,T) \in \bar{\sigma}(\Sigma_i)\}. \tag{46}$$

Proof. Simply note that the recursions in Theorem 8.2 and Corollary 8.8 are identical. □

One corollary to Theorem 1.5 is that you can "hear the shape" of a shifted complex (Corollary 8.9), but only if your ears are fine enough (Remark 8.11).

Corollary 8.9. *A shifted complex Σ^d is completely determined by its spectra $\{\mathbf{s}^{ud}_{i-1}(\Sigma)\}^d_{i=0}$.*

Proof. By Theorem 1.5, the spectra $\{\mathbf{s}^{ud}_{i-1}(\Sigma)\}^d_{i=0}$ determine the z-polynomials $z(S,T)$, and it is easy to see that the long signature (S,T) can be recovered from $z(S,T)$. Furthermore the (short) signature is even more easily recovered from the long signature. When $F = \{v_1 < \ldots < v_i\}$ is a face of Σ that is $\preceq$-maximal, then $(F, F\backslash\{v_i\} \dot\cup \{v_{i+1}\})$ is a critical pair, with (short) signature F. Thus, among all the (short) signatures of Σ, we will find all $\preceq$-maximal faces. Furthermore, every (short) signature is a face of Σ, by definition. Thus, the union $\cup_{F\in\sigma(\Sigma)}\{G\colon G \preceq F\}$ of all $\preceq$-order ideals will yield all the non-empty faces of Σ. □

In fact, if Σ^d is a *pure* shifted complex, then it is determined uniquely by its top Laplacian spectrum $\mathbf{s}^{ud}_{d-1}(\Sigma)$.

Specializing (or "coarsening") the algebraic fine weighting, we obtain as another corollary to Theorem 1.5 Duval and Reiner's description of the unweighted Laplacian eigenvalues of a shifted complex [13, Thm. 1.1], as we now explain.

The *coarse weighting* is obtained from the algebraic fine weighting by omitting all first subscripts, i.e., replacing $x_{i,j} = x_j$ and $X_{i,j} = X_j$. (Thus the monomial corresponding to a facet or set of facets records the degree of each vertex, but forgets the information about the order of vertices in facets.) Note that if $T = [1,t]$, then every z-polynomial $z(S,T)$ specializes to the linear form $E_t = X_1 + \cdots + X_t$ in the coarse weighting.

For a partition λ (a weakly decreasing list of positive integers), let E_λ be the multiset in which each part i of λ is replaced by E_i. Recall that the *conjugate* of λ is the partition λ' in which each part t occurs with multiplicity $\lambda_t - \lambda_{t+1}$.

Corollary 8.10. *Let Σ^d be a shifted simplicial complex on vertices $[n]$. Then*

$$\hat{s}^{ud}_{d-1}(\Sigma) \stackrel{\circ}{=} E_{(\deg_{\Sigma_d})'}$$

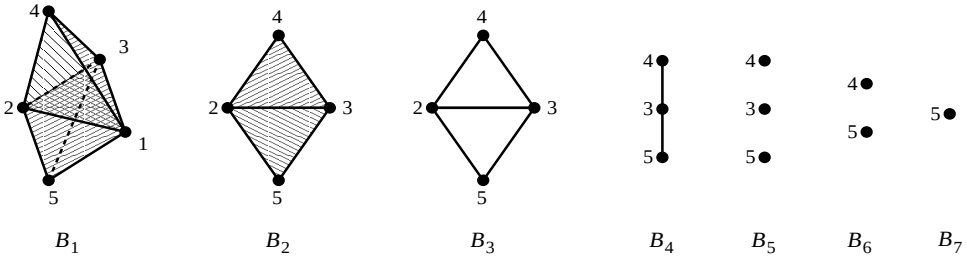
where the left-hand side denotes the multiset of coarsely weighted Laplacian eigenvalues, and the right-hand side is the conjugate of the partition $(\deg_{\Sigma_d}(1), \ldots, \deg_{\Sigma_d}(n))$.

Proof. Theorem 1.5 and the preceding discussion imply that $\hat{s}^{ud}_{d-1}(\Sigma)$ is $\stackrel{\circ}{=}$-equivalent to the multiset in which E_t occurs with multiplicity equal to the number of critical pairs (A, B) of Σ_d such that $t = \max \sigma(A, B)$. By Proposition 8.5, that multiplicity is $\deg_{\Sigma_d}(t) - \deg_{\Sigma_d}(t+1)$. The result now follows from the definition of conjugate partition. □

Passing to the unweighted setting by setting $x_i = 1$ for all i recovers the theorem of Duval and Reiner [13, Thm. 1.1], which states that the Laplacian eigenvalues of a shifted complex Σ are given by the conjugate of the partition $\deg_{\Sigma_d}$.

Remark 8.11. Duval and Reiner also showed [13, Example 10.2] that there are two non-isomorphic 2-dimensional shifted complexes with the same degree sequence. Corollary 8.10 then shows that, in contrast to Corollary 8.9, the coarsely-weighted eigenvalues are not enough to determine a shifted complex.

8.4. **An example: the equatorial bipyramid.** To illustrate Theorems 8.2 and 1.5, we calculate the top-dimensional up-down Laplacian spectrum of the bipyramid $B = \langle 235 \rangle$ (see Examples 1.7, 3.4 and 5.2). In our recursive calculation, we will encounter the subcomplexes $B_2, \ldots, B_7$ of $B = B_1$ shown in the following figure, and B_8, the simplicial complex whose only face is the empty face.



Observe that

- B_1 is a near-cone with apex 1, $\operatorname{del}_1 B_1 = B_2$, and $\operatorname{link}_1 B_1 = B_3$;
- $B_2 = 2 * B_4$;
- B_3 is a near-cone with apex 2, $\operatorname{del}_2 B_3 = B_4$, and $\operatorname{link}_2 B_3 = B_5$;
- $B_4 = 3 * B_6$;
- B_5 is a near-cone with apex 3, $\operatorname{del}_3 B_5 = B_6$, and $\operatorname{link}_3 B_5 = B_8$;
- B_6 is a near-cone with apex 4, $\operatorname{del}_4 B_6 = B_7$, and $\operatorname{link}_4 B_6 = B_8$; and
- $B_7 = 5 * B_8$.

The critical pairs, signatures, and nonzero top-dimensional Laplacian eigenvalues of the bipyramid $B = B_1$ and its subcomplexes $B_2, \ldots, B_7$ are listed in the following table. This information can be obtained either recursively (using Theorem 8.2 repeatedly) or bijectively (from Theorem 1.5).

Subcomplex	Dimension	Vertices	Critical pairs	Signatures	Eigenvalues
B_7	0	$\{5\}$	$(5,6)$	5	$z(\emptyset,5)$
B_6	0	$\{4,5\}$	$(5,6)$	5	$z(\emptyset,45)$
B_5	0	$\{3,4,5\}$	$(5,6)$	5	$z(\emptyset,345)$
B_4	1	$\{3,4,5\}$	$(35,45)$	3	$z(\emptyset,3)$
			$(35,36)$	35	$z(3,345)$
B_3	1	$\{2,3,4,5\}$	$(25,26)$	25	$z(2,2345)$
			$(35,36)$	35	$z(3,2345)$
			$(35,45)$	3	$z(\emptyset,23)$
B_2	2	$\{2,3,4,5\}$	$(235,236)$	235	$z(23,2345)$
			$(235,245)$	23	$z(2,23)$
B_1	2	$\{1,2,3,4,5\}$	$(125,126)$	125	$z(12,12345)$
			$(135,136)$	135	$z(13,12345)$
			$(135,145)$	13	$z(1,123)$
			$(235,236)$	235	$z(23,12345)$
			$(235,245)$	23	$z(2,123)$

We also see from the above table that the coarsely-weighted eigenvalues $\hat{s}_1^{ud}(B_1)$ are $\overset{\circ}{=}$-equivalent to E_5, E_5, E_5, E_3, E_3 (each E_5 coming from a $z(S,12345)$ in B_1, and each E_3 coming from a $z(S,123)$ in B_1), corresponding to the transpose of the degreee sequence of the facets, 55533. (In this case, both the degree sequence and its transpose are 55533.)

9. Enumerating spanning trees of shifted complexes

We now translate Theorem 1.5 from the algebraic to the combinatorial fine weighting, in order to obtain a factorization of the finely weighted spanning tree enumerator $\hat{\tau}(\Sigma)$ of a shifted complex Σ.

Recall that the long signature $\bar{\sigma}(\mathcal{F})$ of a family $\mathcal{F}$ is the multiset of long signatures of its critical pairs.

Theorem 1.6. *Let Σ^d be a shifted complex with initial vertex p. Then:*

$$\hat{\tau}_d(\Sigma) = \left(\prod_{F\in\Lambda_{d-1}} X_{\tilde{F}}\right)\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)} \frac{z(S,\tilde{T})}{X_{1,p}}\right) \tag{47}$$

$$= \left(\prod_{F\in\Lambda_{d-1}} X_{\tilde{F}}\right)\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)} \frac{\sum_{j\in\tilde{T}} X_{S\cup j}}{X_{\tilde{S}}}\right) \tag{48}$$

where $\tilde{F} = F\cup p$, $\Delta = \operatorname{del}_p\Sigma$, and $\Lambda = \operatorname{link}_p\Sigma$.

Proof. Since the complex Σ is APC, its spanning trees are precisely those of its pure d-skeleton $\Sigma_{[d]}$. Similarly, passing from Σ to $\Sigma_{[d]}$ does not affect Δ_d or Λ_{d-1}. Therefore, we may assume without loss of generality that Σ is pure.

By Lemma 6.3 and Theorem 1.5, we have

$$\hat{\tau}_d(\Sigma) = \left(\prod_{F\in\operatorname{link}_p\Sigma} \uparrow X_F\right)\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)} (X_{1,p}+z(S,T))\right) X_{1,p}^m$$

where m is the number of zero eigenvalues of $\mathbf{L}^{\mathrm{ud}}_{\Delta,d-1}$. Since $\mathbf{L}^{\mathrm{ud}}_{\Delta,d-1}$ acts on Δ_{d-1}, but has $\bar{\sigma}(\Delta_d)$ nonzero eigenvalues (including multiplicity), $m = |\Delta_{d-1}| - |\bar{\sigma}(\Delta_d)| = |\Lambda_{d-1}| - |\bar{\sigma}(\Delta_{d-1})|$, since

$\Delta_{d-1} = \Lambda_{d-1}$ by equation (39). Thus,

$$\begin{aligned}\hat{\tau}_d(\Sigma) &= X_{1,p}^{|\Lambda_{d-1}|}\left(\prod_{F\in\Lambda_{d-1}}(\uparrow X_F)\right)X_{1,p}^{-|\bar{\sigma}(\Delta_d)|}\left(\prod_{(S,T)\in\bar{\sigma}(\Delta_d)}X_{1,p}+z(S,T)\right)\\ &= \prod_{F\in\Lambda_{d-1}}(X_{1,p}\uparrow X_F)\prod_{(S,T)\in\bar{\sigma}(\Delta_d)}\frac{X_{1,p}+z(S,T)}{X_{1,p}}.\end{aligned}$$

Equation (47) now follows from equations (19a) and (42). Equation (48) then follows from the definition of z-polynomial (41) and from (19a) again, because

$$\frac{z(S,\tilde{T})}{X_{1,p}} = \frac{1}{X_{1,p}}\frac{\sum_{j\in\tilde{T}}X_{S\cup j}}{\uparrow X_S} = \frac{\sum_{j\in\tilde{T}}X_{S\cup j}}{X_{\tilde{S}}}.$$

□

Example 9.1. We return to our running example, the equatorial bipyramid B. Here $d = 2$ and $p = 1$, and in the notation of Section 8.4, we have $\Delta = B_2$ and $\Lambda = B_3$. Moreover, $\bar{\sigma}(\Delta_d) = \{(2, 23), (23, 2345)\}$. Hence equation (48) yields

$$\hat{\tau}(B) = X_{123}X_{124}X_{134}X_{125}X_{135}\left(\frac{X_{12}+X_{22}+X_{23}}{X_{12}}\right)\left(\frac{X_{123}+X_{223}+X_{233}+X_{234}+X_{235}}{X_{123}}\right).$$

Note that this is a genuine polynomial (not just a Laurent polynomial) in the indeterminates $X_{i,j}$.

Corollary 9.2. *Let Σ^d be a shifted complex with initial vertex 1. Let $\Delta = \operatorname{del}_1\Sigma$, $\tilde{\Delta} = 1 * \Delta$, $\Lambda = \operatorname{link}_1\Sigma$, and $\tilde{\Lambda} = 1 * \Lambda$. Then, in the coarse weighting,*

$$\hat{\tau}_d(\Sigma) = X^{\deg\tilde{\Lambda}_d}\prod_i (E_i/X_1)^{(\deg\tilde{\Delta}_{d+1})'_i},$$

where $E_i = X_1 + \cdots + X_i$ and, for a partition λ, we set $X^\lambda := \prod_i X_i^{\lambda_i}$.

Proof. Upon coarsening the weighting, the first product in (48) becomes $X^{\deg\tilde{\Lambda}_d}$ and the second product becomes

$$\prod_{(S,T)\in\bar{\sigma}(\Delta_d)}\frac{\sum_{j\in\tilde{T}}X_j}{X_1} = \prod_{(S,T)\in\bar{\sigma}(\Delta_d)}E_{|T|+1}/X_1.$$

We now claim that

$$\prod_{(S,T)\in\bar{\sigma}(\Delta_d)}E_{|T|+1}/X_1 = \prod_{(S,T)\in\bar{\sigma}(\tilde{\Delta}_{d+1})}E_{|T|}/X_1.$$

Indeed, by Proposition 8.7,

$$\sigma(\tilde{\Delta}_{d+1}) = \{1\,\dot\cup\,F\colon F\in\sigma(\Delta_d)\}\cup\{1\}^m$$

for some m, since $\Delta = \operatorname{link}_1(\tilde{\Delta})$ and $\dim(\operatorname{del}_1(\tilde{\Delta})) = \dim\Delta < d+1$. It now follows that

$$\prod_{(S,T)\in\bar{\sigma}(\tilde{\Delta}_{d+1})}E_{|T|}/X_1 = \prod_{(S,T)\in\bar{\sigma}(\Delta_d)}(E_{|T|+1}/X_1)(X_1/X_1)^m,$$

implying the claim. Finally, by Proposition 8.5 and the definition of conjugate partition, we obtain

$$\prod_{(S,T)\in\bar{\sigma}(\tilde{\Delta}_{d+1})}E_{|T|}/X_1 = \prod_t (E_t/X_1)^{\deg_{\tilde{\Delta}_{d+1}}(t)-\deg_{\tilde{\Delta}_{d+1}}(t+1)} = \prod_i (E_i/X_1)^{(\deg\tilde{\Delta}_{d+1})'_i}.$$

□

Example 9.3. Once again, let B be the equatorial bipyramid. Here $\tilde{\Lambda}_d = \{123, 124, 125, 134, 135\}$. Its degree sequence is 53322. Thus the monomial factor in Corollary 9.2 is

$$X_{123}X_{124}X_{125}X_{134}X_{135} = X_1^5X_2^3X_3^3X_4^2X_5^2.$$

Meanwhile, $\tilde{\Delta}_{d+1} = \{1234, 1235\}$. Its degree sequence is 22211, with conjugate 53. The product factor in Corollary 9.2 is therefore

$$(E_5/X_1)(E_3/X_1) = (X_1 + X_2 + X_3 + X_4 + X_5)(X_1 + X_2 + X_3)/X_1^2.$$

Putting these terms together yields

$$\hat{\tau}_d(\Sigma) = X_1^3X_2^3X_3^3X_4^2X_5^2(X_1 + X_2 + X_3 + X_4 + X_5)(X_1 + X_2 + X_3)$$

which matches Example 5.2.

10. Corollaries

We conclude by showing how several known tree enumerators—for skeletons of simplices, threshold graphs, and Ferrers graphs—can be recovered from our results.

10.1. **Skeletons of simplices.** Let Σ be the d-skeleton of the simplex on vertices $[n]$, so that the set of facets of Σ is $\binom{[n]}{d+1}$, the set of all subsets of $[n]$ of cardinality $d+1$. Note that Σ is generated as a shifted complex by the single facet $[n-d, n]$. The critical pairs of Σ are

$$\left\{(A \cup \{n\},\ A \cup \{n+1\})\colon A \in \binom{[n-1]}{d}\right\}$$

and the corresponding long signatures are

$$\bar{\sigma}(\Sigma_d) = \left\{(A,\ [n])\colon A \in \binom{[n-1]}{d}\right\}.$$

Setting $\Lambda = \operatorname{link}_1 \Sigma$ and $\Delta = \operatorname{del}_1 \Sigma$, we have

$$\Lambda_{d-1} = \binom{[2,n]}{d};$$
$$\Delta_d = \binom{[2,n]}{d+1};$$
$$\bar{\sigma}(\Delta_d) = \left\{(B,\ [2,n])\colon B \in \binom{[2,n-1]}{d}\right\}.$$

Applying equation (48), we obtain

$$\hat{\tau}_d(\Sigma) = \left(\prod_{\substack{C \subseteq [2,n] \\ |C|=d}} X_{\tilde{C}}\right)\left(\prod_{\substack{B \subseteq [2,n-1] \\ |B|=d}} \frac{\sum_{j=1}^n X_{B\cup j}}{X_{\tilde{B}}}\right).$$

The denominators in the second product cancel the factors $X_{\tilde{C}}$ in the first product with $n \notin \tilde{C}$, leaving only those for which $n \in \tilde{C}$. Therefore,

$$\hat{\tau}_d(\Sigma) = \left(\prod_{\substack{C \subseteq [2,n] \\ n \in C \\ |C|=d}} X_{\tilde{C}}\right)\left(\prod_{\substack{B \subseteq [2,n-1] \\ |B|=d}} \sum_{j=1}^n X_{B\cup j}\right) \quad = \left(\prod_{\substack{C \subseteq [n] \\ 1,n \in C \\ |C|=d+1}} X_C\right)\left(\prod_{\substack{B \subseteq [2,n-1] \\ |B|=d}} z(B,[n])\right).$$

Passing to the coarse weighting by setting $X_{i,j} = X_j$ for every i, j, we obtain

$$\begin{aligned}\hat{\tau}_d(\Sigma) &= \Big((X_1X_n)^{\binom{n-2}{d-1}}(X_2\cdots X_{n-1})^{\binom{n-3}{d-2}}\Big)\Big((X_2\cdots X_{n-1})^{\binom{n-3}{d-1}}(X_1+\cdots+X_n)^{\binom{n-2}{d}}\Big)\\ &= (X_1\cdots X_n)^{\binom{n-2}{d-1}}(X_1+\cdots+X_n)^{\binom{n-2}{d}},\end{aligned}$$

which is Theorem 3′ of Kalai's paper [20]. Furthermore, setting $X_i = 1$ for all i recovers Kalai's generalization of Cayley's formula: $\tau_d(\Sigma) = n^{\binom{n-2}{d}}$.

10.2. **Threshold graphs.** A *threshold graph* is a one-dimensional shifted complex Σ. For simplicity, we assume that the vertex set of Σ is $[1, n]$. We may also assume that Σ is connected, so that every vertex is adjacent to vertex 1. Martin and Reiner [27, Theorem 4] found a factorization of the combinatorially finely weighted spanning tree enumerator of Σ, which may be stated in our notation[5] as:

$$\hat{\tau}(\Sigma) = X_{\{1,n\}}\prod_{v=2}^{n-1}\sum_{j=1}^{(\deg\Sigma)'_v} X_{\{v,j\}}. \tag{49}$$

A somewhat more general result was obtained independently by Remmel and Williamson [31, Theorem 2.4]. We will show how this formula can be recovered from Theorem 1.6.

The first product in equation (48) is just $X_{\{1,2\}}X_{\{1,3\}}\cdots X_{\{1,n\}}$. For the second product, we must identify the critical pairs of $\Delta = \operatorname{del}_1\Sigma$. Note that Δ is a threshold graph with vertices $[2, n]$.

As is often the case with threshold graphs (and their degree sequences, which we will soon encounter), we need to sort vertices by their relation to the size of the *Durfee square* of Σ, the largest square that fits in the Ferrers diagram of its degree sequence. The side length m of the Durfee square is the largest number such that $\{m, m+1\}$ is an edge of Σ. If $m = 1$, then Σ is a star graph, the Ferrers diagram of its degree sequence is a hook, and equations (48) and (49) both easily reduce to

$$\hat{\tau}_d(\Sigma) = \prod_{v=2}^{n} X_{\{1,v\}}.$$

Therefore, we henceforth assume that $m \geq 2$. Note that every edge has at least one endpoint $\leq m$ (because $\{m+1, m+2\} \notin \Sigma$, and that edge is the unique $\preceq$-minimal edge with both endpoints $> m$).

For each vertex v of Σ, let

$$w(v) = \max\{u\colon \{u, v\} \in \Sigma\}.$$

Note that if $v \leq m$, then $\{v, m+1\} \preceq \{m, m+1\} \in \Sigma$, so $w(v) \geq m$. On the other hand, if $v > m$, then $w(v) < v$.

Lemma 10.1. *The critical pairs of $\Delta = \operatorname{del}_1\Sigma$ are as follows.*

For each $v_1 \in [2, m]$, there is a "type I" critical pair $(\{v_1, v_2\}, \{v_1, v_2+1\})$ where $v_2 = w(v_1)$.

For each $v_2 \in [m+2, w(2)]$, there is a "type II" critical pair $(\{v_1, v_2\}, \{v_1+1, v_2\})$, where $v_1 = w(v_2)$.

Furthermore, every critical pair is of one of these forms.

Proof. It is immediate from the definition of $w(v)$ that each such pair is critical. Suppose now that (A, B) is a critical pair of Δ, with $A = \{v_1 < v_2\}$. Then either $B = \{v_1, v_2+1\}$ or $B = \{v_1+1, v_2\}$.

If $B = \{v_1, v_2+1\}$, then we have already observed that A has at least one endpoint in $[2, m]$. In particular, $v_1 \leq m$, and the pair (A, B) is of type I.

If $B = \{v_1+1, v_2\}$, then $A \in \Delta$ and $B \notin \Delta$, so by definition $v_1 = w(v_2)$. Moreover, $m+2 \leq v_2$ (because $v_1 + 1 \leq v_2 - 1$, so $\{v_2 - 1, v_2\} \succeq B \notin \Delta$) and $v_2 \leq w(2)$ (because $\{2, v_2\} \preceq A$, so $\{2, v_2\} \in \Delta$). Hence the pair (A, B) is of type II. □

[5]Note the distinction between the variable $X_{i,j}$, which corresponds to vertex j as the ith smallest vertex in a face, and the quadratic monomial $X_{\{i,j\}}$, which corresponds to the edge $\{i, j\}$, and which equals $X_{1,i}X_{2,j}$ if $i \leq j$ but equals $X_{1,j}X_{2,i}$ if $j \leq i$.

If (A, B) is a critical pair of type I, then $\sigma(A, B) = \{v_1, v_2\}$ and $\bar{\sigma}(A, B) = (\{v_1\}, [2, v_2])$. If (A, B) is a critical pair of type II, then $\sigma(A, B) = \{v_1\}$ and $\bar{\sigma}(A, B) = (\emptyset, [2, v_1])$. Therefore, formula (48) yields

$$\begin{aligned}\hat{\tau}_d(\Sigma) &= \prod_{v=2}^{n} X_{\{1,v\}} \prod_{v_1=2}^{m} \frac{\sum_{j=1}^{w(v_1)} X_{\{v_1,j\}}}{X_{\{1,v_1\}}} \prod_{v_2=m+2}^{w(2)} \frac{\sum_{j=1}^{w(v_2)} X_{\{j\}}}{X_{\{1\}}} \\ &= \prod_{v=2}^{n} X_{\{1,v\}} \prod_{v_1=2}^{m} \frac{\sum_{j=1}^{w(v_1)} X_{\{v_1,j\}}}{X_{\{1,v_1\}}} \prod_{v_2=m+2}^{n} \frac{\sum_{j=1}^{w(v_2)} X_{\{j\}}}{X_{\{1\}}} \\ &= X_{\{1,n\}} \left(\prod_{v_1=2}^{m} X_{\{1,v_1\}} \frac{\sum_{j=1}^{w(v_1)} X_{\{v_1,j\}}}{X_{\{1,v_1\}}} \right) \left(\prod_{v_2=m+2}^{n} X_{\{1,v_2-1\}} \frac{\sum_{j=1}^{w(v_2)} X_{\{j\}}}{X_{\{1\}}} \right). \end{aligned} \tag{50}$$

The second equality follows because $w(v) = 1$ whenever $v > w(2)$, and the third equality comes from redistributing most of the first product among the other two. Now, when $v_2 > m+1$, we have

$$X_{\{1,v_2-1\}} \frac{\sum_{j=1}^{w(v_2)} X_{\{j\}}}{X_{\{1\}}} = \frac{X_{1,1} X_{2,v_2-1}}{X_{1,1}} \sum_{j=1}^{w(v_2)} X_{1,j} = \sum_{j=1}^{w(v_2)} X_{1,j} X_{2,v_2-1} = \sum_{j=1}^{w(v_2)} X_{\{j,v_2-1\}}$$

since $j \le w(v_2) \le m < v_2 - 1$. Thus we may rewrite (50) as

$$\hat{\tau}_d(\Sigma) = \left(\prod_{v_1=2}^{m} \sum_{j=1}^{w(v_1)} X_{\{v_1,j\}} \right) \left(\prod_{v_2=m+2}^{n} \sum_{j=1}^{w(v_2)} X_{\{j,v_2-1\}} \right) X_{\{1,n\}}. \tag{51}$$

If $v_1 \le m$, then $w(v_1) > v_1$, so $w(v_1)$ has degree at least v_1, as does every vertex less than $w(v_1)$. On the other hand, $\{v_1, w(v_1)+1\} \notin \Sigma$, so vertex $w(v_1)+1$ has degree less than v_1, as does every vertex greater than $w(v_1)+1$. Therefore, $(\deg \Sigma)'_{v_1} = w(v_1)$.

Similarly, if $v_2 > m+1$, then $(\{w(v_2), v_2)\}, \{w(v_2)+1, v_2\})$ is a critical pair, so $w(v_2)$ has degree at least $v_2 - 1$, as does every vertex less than $w(v_2)$. On the other hand, vertex $w(v_2)+1$ has degree less than $v_2 - 1$, as does every vertex greater than $w(v_2)+1$. Therefore, $(\deg \Sigma)'_{v_2-1} = w(v_2)$.

Using these observations to rewrite (51) in terms of the partition $(\deg \Sigma)'$ recovers the Martin-Reiner formula (49).

10.3. **From threshold graphs to Ferrers graphs.** Let $\lambda = (\lambda_1 \ge \cdots \ge \lambda_\ell)$ be a partition. The corresponding *Ferrers graph* is the bipartite graph with vertices $x_1, \ldots, x_{\lambda_1}, y_1, \ldots, y_\ell$ and edges $\{x_i y_j : i \le \lambda_j\}$. That is, the vertices correspond to rows and columns of the Ferrers diagram of λ, and the edges to squares appearing in the diagram. Ehrenborg and van Willigenburg [14] considered Ferrers graphs and (among other results) described how a certain weighted spanning tree enumerator splits into linear factors. Another proof of their formula can be obtained from the foregoing formulas for threshold graphs, as we now explain. The key idea is due to Richard Ehrenborg.

Let G be a connected threshold graph on vertices $[n]$, and let m be the side length of the Durfee square of G. Then the vertices $1, 2, \ldots, m$ are pairwise adjacent, while $m+1, \ldots, n$ are pairwise nonadjacent. Moreover, if $m+1 \le i < j \le n$, then every neighbor of j is a neighbor of i. Construct a graph F by deleting all edges ij such that $i, j \le m$. Then F is a Ferrers graph; furthermore, all Ferrers graphs can be constructed in this way. Thus, if we begin with the weighted enumerator for G and set to zero all indeterminates corresponding to edges between vertices $1, 2, \ldots, m$, we recover the weighted enumerator for the corresponding Ferrers graph F. Specifically, Theorem 1.6 yields

$$\hat{\tau}(G) = X_{1,1} X_{n,2} \prod_{i=2}^{n-1} \left(\sum_{r=1}^{d_i^T(G)} X_{\min(i,r),1} X_{\max(i,r),2} \right) \tag{52}$$

(this is also [27, Theorem 4]). Breaking up the product in (52) around the parameter m gives

$$\tau(G) = X_{1,1}X_{n,2}\left(\prod_{i=2}^{m}\left(\sum_{r=1}^{m} X_{\min(i,r),1}X_{\max(i,r),2} + \sum_{r=m+1}^{d_i^T(G)} X_{\min(i,r),1}X_{\max(i,r),2}\right)\right) \times \left(\prod_{i=m+1}^{n-1}\sum_{r=1}^{d_i^T(G)} X_{\min(i,r),1}X_{\max(i,r),2}\right).$$

This expression is well defined because $d_i^T(G) \geq m$ whenever $i \leq m$. If $i \leq m$ and $r \leq m$, then $\max(i,r) \leq m$, and these are exactly the terms we wish to set to zero. Therefore,

$$\tau(F) = X_{1,1}X_{n,2}\left(\prod_{i=2}^{m}\sum_{r=m+1}^{d_i^T(G)} X_{i,1}X_{r,2}\right)\left(\prod_{i=m+1}^{n-1}\sum_{r=1}^{d_i^T(G)} X_{\min(i,r),1}X_{\max(i,r),2}\right). \tag{53}$$

For $i \geq m+1$, we have $d_i^T(G) < m < i$. Thus $r < i$ for $r \leq d_i^T(G)$, and (53) yields

$$\begin{aligned}\tau(F) &= X_{1,1}X_{n,2}\left(\prod_{i=2}^{m} X_{i,1}\prod_{i=2}^{m}\sum_{r=m+1}^{d_i^T(G)} X_{r,2}\right)\left(\prod_{i=m+1}^{n-1} X_{i,2}\prod_{i=m+1}^{n-1}\sum_{r=1}^{d_i^T(G)} X_{r,1}\right)\\ &= (X_{1,1}X_{2,1}\dots X_{m,1})(X_{m+1,2}X_{m+2,2}\dots X_{n,2})\left(\prod_{i=2}^{m}\sum_{r=m+1}^{d_i^T(G)} X_{r,2}\right)\left(\prod_{i=m+1}^{n-1}\sum_{r=1}^{d_i^T(G)} X_{r,1}\right). \end{aligned}\tag{54}$$

By construction, the vertex degrees in G and F are related by the formula

$$\deg_F(i) = \begin{cases}\deg_G(i) - m & \text{if } 1 \leq i \leq m,\\ \deg_G(i) & \text{if } m+1 \leq i \leq n.\end{cases}$$

To simplify the notation, set $X_{r,1} = x_r$ and $y_{r-m} = X_{r,2} = y_{r-m}$. (From this perspective, the two partite sets of F correspond to the indeterminates $\{x_1, \dots x_m\}$ and $\{y_1, \dots y_{n-m}\}$. Therefore,

$$\tau(F) = (x_1 \dots x_m)(y_1 \dots y_{n-m})\left(\prod_{i=2}^{m}\sum_{r=1}^{d_i^T(F_2)} y_r\right)\left(\prod_{i=2}^{n-m}\sum_{r=1}^{d_i^T(F_1)} x_r\right)$$

which is Theorem 2.1 of [14].

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Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968-0514

Departments of Mathematics and Computer Science, The University of Chicago, Chicago, IL 60637

Department of Mathematics, University of Kansas, Lawrence, KS 66047