A NON-PARTITIONABLE COHEN–MACAULAY SIMPLICIAL COMPLEX

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Abstract. A long-standing conjecture of Stanley states that every Cohen–Macaulay simplicial complex is partitionable. We disprove the conjecture by constructing an explicit counterexample. Due to a result of Herzog, Jahan and Yassemi, our construction also disproves the conjecture that the Stanley depth of a monomial ideal is always at least its depth.

1. Introduction

Cohen–Macaulay simplicial complexes are ubiquitous in algebraic and topological combinatorics. They were introduced in 1975 by Stanley in his celebrated proof of the Upper Bound Conjecture for spheres [Sta75b].[1] The theory of Cohen–Macaulay *rings* has long been of great importance in algebra and algebraic geometry; see, e.g., [Ree57, ZS60, Gro64, Hoc72, Hoc80, BH93]. The connection to combinatorics via what is known as Stanley–Reisner theory was established by Hochster [Hoc72], Reisner [Rei76], and Stanley [Sta75a]; standard references for this subject are [Sta96] and [BH93].

The focus of this article is the following conjecture, described by Stanley as "a central combinatorial conjecture on Cohen–Macaulay complexes" [Sta96, p. 85]. It was originally proposed by Stanley [Sta79, p. 149] in 1979 and independently by Garsia [Gar80, Remark 5.2] in 1980 for order complexes of Cohen–Macaulay posets.

Conjecture 1.1 (Partitionability Conjecture)**.** *Every Cohen–Macaulay simplicial complex is partitionable.*

We explicitly construct a Cohen–Macaulay complex that is not partitionable, thus disproving the Partitionability Conjecture. In fact, we give a general method for constructing counterexamples and an explicit infinite family of non-partitionable Cohen–Macaulay complexes. We begin by giving some background for the conjecture, which will also be directly relevant in our construction.

Two basic invariants of a simplicial complex Δ are its f- and h-vectors

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_d(\Delta)), \qquad h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_{d+1}(\Delta)),$$

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[1] See [Sta14] for his engaging and personal account of how the proof came to be.

where $d = \dim \Delta$. The number $f_i = f_i(\Delta)$ is the number of i-dimensional faces (simplices) in Δ. The h-vector is more subtle. It carries the same information as the f-vector (the two are related by an invertible linear transformation), and arises naturally in algebra: the Hilbert series of the Stanley–Reisner ring of Δ is $(1-t)^{-d} \sum_j h_j(\Delta) t^j$. (See Section 2 for precise definitions.) It is not at all apparent if the numbers $h_j(\Delta)$ have a combinatorial interpretation; for instance, they need not be positive in general.

A *partitioning* of a pure simplicial complex Δ is a decomposition into pairwise-disjoint Boolean intervals whose maximal elements are exactly the facets (maximal faces) of Δ. Partitionability was introduced by Provan [Pro77] and Ball [Bal77] in the context of reliability analysis. For a partitionable complex, the h-numbers enumerate the minimum elements of the intervals by size. In particular, shellable complexes are easily seen to be partitionable, and hence their h-vectors have this interpretation. The strict inclusions

$$\{\text{shellable complexes}\} \subsetneq \{\text{constructible complexes}\} \subsetneq \{\text{Cohen–Macaulay complexes}\}$$

are also well known. For example, the nonshellable balls constructed by Rudin [Rud58] and Ziegler [Zie98] are constructible (see also [Lut04a]), and any triangulation of the dunce hat is Cohen–Macaulay but not constructible [Hac08, §2]. On the other hand, the possible h-vectors of Cohen–Macaulay, constructible, and shellable complexes are all the same [Sta77, Theorem 6], suggesting that their entries ought to count something explicit. The Partitionability Conjecture would have provided a combinatorial interpretation of the h-vectors of Cohen–Macaulay complexes.

The idea of our construction is to work with *relative* simplicial complexes. Suppose $Q = (X, A)$ is a relative simplicial complex that is not partitionable, but with X and A Cohen–Macaulay. Theorem 3.1 gives a general method of gluing together sufficiently many copies of X along A to obtain a counterexample to the Partitionability Conjecture, provided that A is an induced subcomplex of X. This reduces the problem to finding an appropriate pair (X, A). Our starting point is the nonshellable simplicial 3-ball Z constructed by Ziegler [Zie98], in which we find a suitable subcomplex A and in turn the desired relative complex Q (Theorem 3.3). By refining the construction, we are able to obtain, in Theorem 3.5, a Cohen–Macaulay non-partitionable complex that is much smaller than predicted by Theorem 3.1, with f-vector $(1, 16, 71, 98, 42)$ and h-vector $(1, 12, 29)$.

The existence of a Cohen–Macaulay nonpartitionable complex has an important consequence in commutative algebra. For a polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ and a $\mathbb{Z}^n$-graded S-module M, many fundamental algebraic invariants of M, such as its dimension and multigraded Hilbert series, can be profitably studied using combinatorics. On the other hand, the combinatorial properties of the *depth* of M are less well understood. In [Sta82], Stanley proposed a purely combinatorial analogue of depth, defined in terms of certain vector space decompositions of M. This invariant, now known as the *Stanley depth* and written $\operatorname{sdepth} M$, has attracted considerable recent attention (see [PSFTY09] for an accessible introduction to the subject, and [Her13] for a comprehensive survey), centering around the following conjecture of Stanley [Sta82, Conjecture 5.1]:

Conjecture 1.2 (Depth Conjecture)**.** *For all $\mathbb{Z}^n$-graded S-modules M,*

$$\operatorname{sdepth} M \geq \operatorname{depth} M.$$

Herzog, Jahan, and Yassemi proved [HJY08, Corollary 4.5] that when I is the Stanley–Reisner ideal of a Cohen–Macaulay complex Δ, the inequality $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$ is equivalent to the partitionability of Δ. Therefore, our counterexample to the Partitionability Conjecture disproves the Depth Conjecture as well. We exhibit a smaller counterexample to the Depth Conjecture using a relative complex in Remark 3.6; see Section 3.2.

It was also previously not known whether all constructible complexes were partitionable; see, e.g., [Hac00, §4]. The counterexample we obtain is not only Cohen–Macaulay, but in fact constructible. Therefore, even constructibility does not imply partitionability.

2. Preliminaries

2.1. **Simplicial and relative simplicial complexes.** Throughout the paper, all complexes will be finite. Let V be a finite set. A *simplicial complex* on V is a collection Δ of subsets of V such that whenever $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. Equivalently, Δ is an order ideal in the Boolean poset 2^V. The symbol $|\Delta|$ denotes the standard geometric realization of Δ. The elements of Δ are called the *faces* of Δ, and the elements of V are *vertices*. Maximal faces are called *facets*. The *dimension* of a face σ is $\dim \sigma = |\sigma| - 1$, and the dimension of Δ is $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$. We often write Δ^d to indicate that $\dim \Delta = d$. A complex is *pure* if all maximal faces have the same dimension. A *subcomplex* of Δ is a simplicial complex Γ with $\Gamma \subseteq \Delta$. A subcomplex is an *induced subcomplex* if it is of the form

$$\Delta|_W := \{\sigma \in \Delta \mid \sigma \subseteq W\}$$

for some $W \subseteq V$.

In the construction of our counterexample, we will work with the more general class of *relative simplicial complexes*. A relative complex Φ on V is a subset of 2^V that is *convex*: if $\rho, \tau \in \Phi$ and $\rho \subseteq \sigma \subseteq \tau$, then $\sigma \in \Phi$. We sometimes refer to simplicial complexes as "absolute" to distinguish them from relative complexes.

Every relative complex can be expressed as a pair $\Phi = (\Delta, \Gamma) := \Delta \setminus \Gamma$, where Δ is a simplicial complex and $\Gamma \subseteq \Delta$ is a subcomplex. Topologically, Φ corresponds to the quotient space $|\Delta|/|\Gamma|$. Note that there are infinitely many possibilities for the pair Δ, Γ. The unique minimal expression is obtained by letting $\Delta = \bar{\Phi}$ be the *combinatorial closure* of Φ, i.e., the smallest simplicial complex containing Φ as a subset, and setting $\Gamma = \Delta \setminus \Phi$. Note that in this case $\dim \Gamma < \dim \Delta$, because the maximal faces of Δ are precisely those of Φ.

The notation $\tilde{H}_i(\Delta)$ denotes the i^{th} reduced simplicial homology group with coefficients in $\mathbb{Z}$. (The underlying ring does not matter for our purposes.) The simplicial homology groups $\tilde{H}_i(\Phi)$ of a relative complex $\Phi = (\Delta, \Gamma)$ are just the relative homology groups $\tilde{H}_i(\Delta, \Gamma)$ in the usual topological sense (see, e.g., [Hat02]); in particular, the homology groups of Δ, Γ, and Φ fit into a long exact sequence.

The *f-vector* of an (absolute or relative) complex Δ^d is $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$, where $f_i = f_i(\Delta)$ is the number of i-dimensional faces of Δ. Note that $f_{-1}(\Delta) = 1$ for every absolute complex other than the void complex $\Delta = \emptyset$. The *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$ is defined by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1}, \qquad 0 \leq k \leq d+1.$$

In particular, the f- and h-vectors determine each other.

The *link* of a face $\sigma \in \Delta$ is defined as

$$\operatorname{link}_\Delta(\sigma) := \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Delta\}.$$

Observe that if Δ^d is pure and $\dim \sigma = k$, then $\dim \operatorname{link}_\Delta(\sigma) = d - k - 1$. If σ is a facet of Δ then $\operatorname{link}_\Delta(\sigma) = \{\emptyset\}$, the trivial complex with only the empty face, and if $\sigma \notin \Delta$ then we set $\operatorname{link}_\Delta(\sigma)$ to be the void complex with no faces.

If $\Phi = (\Delta, \Gamma)$ is a relative complex and $\sigma \in \Delta$, we can define the relative link by

$$\operatorname{link}_\Phi(\sigma) \ = \ (\operatorname{link}_\Delta(\sigma), \operatorname{link}_\Gamma(\sigma)).$$

It is easy to check that this construction is intrinsic to Φ, i.e., it does not depend on the choice of the pair Δ, Γ. Note that $\operatorname{link}_\Phi(\sigma)$ is not necessarily a subset of Φ.

2.2. **Cohen–Macaulay simplicial complexes.** A ring is *Cohen–Macaulay* if its depth equals its (Krull) dimension. Reisner's criterion [Rei76, Theorem 1] states that Cohen–Macaulayness of the Stanley–Reisner ring [Sta96, §II.1] of a simplicial complex can be expressed in terms of simplicial homology, and we will take this criterion as our definition. The relative version of Reisner's criterion is Theorem 5.3 of [Sta87].

Theorem 2.1. [Rei76, Sta87] *A simplicial complex Δ is* Cohen–Macaulay *if for every face $\sigma \in \Delta$,*

$$\tilde{H}_i(\operatorname{link}_\Delta(\sigma)) = 0 \quad \text{for } i < \dim \operatorname{link}_\Delta(\sigma). \tag{1}$$

Similarly, a relative complex $\Phi = (\Delta, \Gamma)$ is Cohen–Macaulay *if for every $\sigma \in \Delta$,*

$$\tilde{H}_i(\operatorname{link}_\Delta(\sigma), \operatorname{link}_\Gamma(\sigma)) = 0 \quad \text{for } i < \dim \operatorname{link}_\Delta(\sigma).$$

In fact, Cohen–Macaulayness is a *topological* invariant: it depends only on the homeomorphism type of the geometric realization $|\Delta|$. This was proved by Munkres [Mun84]. Topological invariance holds for relative complexes as well [Sta96, Corollary III.7.3]. Importantly, if $|\Delta|$ is homeomorphic to a ball or to a sphere, then Δ is Cohen–Macaulay [Mun84, §2].

The following technical lemma will be central to our construction.

Lemma 2.2. *Let Δ_1 and Δ_2 be d-dimensional Cohen–Macaulay simplicial complexes on disjoint vertex sets. Let Γ be a Cohen–Macaulay simplicial complex of dimension d or $d-1$, and suppose that each Δ_i contains a copy of Γ as an induced subcomplex. Then the complex Ω obtained by identifying the two copies of Γ (or "gluing together Δ_1 and Δ_2 along Γ") is Cohen–Macaulay.*

Proof. It is clear that Ω is a CW-complex. The requirement that each copy of Γ is an *induced* subcomplex of Δ_i means that Ω is in fact a simplicial complex (because faces with the same underlying vertex set will be identified). It remains to show that Ω is Cohen–Macaulay. Henceforth, to simplify the notation, we will identify Γ with $\Delta_1 \cap \Delta_2$, so that Ω is identified with $\Delta_1 \cup \Delta_2$.

Let σ be a face of Ω. Note that

$$\begin{aligned} \operatorname{link}_\Omega(\sigma) &= \{\tau \in \Omega \mid \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Omega\} = \operatorname{link}_{\Delta_1}(\sigma) \cup \operatorname{link}_{\Delta_2}(\sigma), \\ \operatorname{link}_\Gamma(\sigma) &= \{\tau \in \Gamma \mid \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Gamma\} = \operatorname{link}_{\Delta_1}(\sigma) \cap \operatorname{link}_{\Delta_2}(\sigma). \end{aligned} \tag{2}$$

First, suppose that $\sigma \in \Delta_1 \setminus \Delta_2$. Then Reisner's criterion (1) holds for σ because $\operatorname{link}_\Omega(\sigma) = \operatorname{link}_{\Delta_1}(\sigma)$, and Δ_1 is Cohen–Macaulay. Likewise, Reisner's criterion holds for faces of $\Delta_2 \setminus \Delta_1$.

On the other hand, suppose that $\sigma \in \Gamma$. Then the observations (2) give rise to a reduced Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(\mathrm{link}_{\Delta_1}(\sigma)) \oplus \tilde{H}_i(\mathrm{link}_{\Delta_2}(\sigma)) \to \tilde{H}_i(\mathrm{link}_{\Omega}(\sigma)) \to \tilde{H}_{i-1}(\mathrm{link}_{\Gamma}(\sigma)) \to \cdots .$$

But since $\Delta_1, \Delta_2, \Gamma$ are Cohen–Macaulay and $\dim \Gamma \geq \dim \Delta_1 - 1$, the Mayer-Vietoris sequence implies that $\tilde{H}_i(\mathrm{link}_{\Omega}(\sigma)) = 0$ for all $i < d - \dim \sigma - 1$. This is precisely the statement that Reisner's criterion holds for σ. □

Iterating Lemma 2.2, we obtain immediately:

Proposition 2.3. *Let $\Delta_1, \ldots, \Delta_n$ be d-dimensional Cohen–Macaulay simplicial complexes on disjoint vertex sets. Let Γ be a Cohen–Macaulay simplicial complex of dimension $d-1$ or d, and suppose that each Δ_i contains a copy of Γ as an induced subcomplex. Then the complex Ω obtained from $\Delta_1, \ldots, \Delta_n$ by identifying the n copies of Γ is Cohen–Macaulay.*

2.3. **Shellability, partitionability, and constructibility.**

Definition 2.4. Let Δ be a pure simplicial complex. A *shelling* of Δ is a total ordering $F_1, \ldots, F_n$ of its facets so that for every j, the set

$$\{\sigma \subseteq F_j \mid \sigma \not\subseteq F_i \text{ for all } i < j\}$$

has a unique minimal element R_j.

The h-vector of a shellable complex has a simple combinatorial interpretation:

$$h_k(\Delta) = \#\{j \mid \#R_j = k\}. \tag{3}$$

In particular $h_k(\Delta) \geq 0$ for all k, and in fact $h_k(\Delta) = 0$ implies $h_\ell(\Delta) = 0$ for all $\ell > k$ (a consequence of [BH93, Theorem 5.1.15]). Shellable complexes are Cohen–Macaulay, although the converse is not true: well-known counterexamples include any triangulation of the dunce hat, as well as the nonshellable balls constructed by Rudin [Rud58] and Ziegler [Zie98]. On the other hand, Cohen–Macaulay complexes satisfy the same conditions on the h-vector, so it is natural to look for a combinatorial interpretation of their h-vectors.

Definition 2.5. Let Δ be a pure simplicial complex with facets $F_1, \ldots, F_n$. A *partitioning* $\mathcal{P}$ of Δ is a decomposition into pairwise-disjoint Boolean intervals

$$\Delta = \bigsqcup_{i=1}^{n} [R_i, F_i]$$

where $[R_i, F_i] = \{\sigma \in \Delta \mid R_i \subseteq \sigma \subseteq F_i\}$. We say that each F_i is *matched* to the corresponding R_i.

Clearly, shellable complexes are partitionable. If Δ is partitionable, then its h-vector automatically carries the combinatorial interpretation (3) [Sta96, Proposition III.2.3]. Moreover, Definitions 2.4 and 2.5, and the interpretation of the h-vector, carry over precisely from absolute to relative complexes.

Example 2.6. A partitionable complex need not be Cohen–Macaulay, much less shellable. The following example is due to Björner [Sta96, p. 85]. Let Δ be the pure 2-dimensional complex with facets $123, 124, 134, 234, 156$ (abbreviating $\{1,2,3\}$ by 123, etc.). This complex is not Cohen–Macaulay because vertex 1 fails Reisner's criterion (1), but it is partitionable:

$$\Delta = [\emptyset, 156] \cup [2, 123] \cup [3, 134] \cup [4, 124] \cup [234, 234].$$

In particular, $h(\Delta) = (1, 3, 0, 1)$, which is not the h-vector of any Cohen–Macaulay complex (since $h_2 = 0$ and $h_3 > 0$).

Constructibility, introduced by Hochster [Hoc72], is a combinatorial condition intermediate between shellability and Cohen–Macaulayness.

Definition 2.7. A complex Δ^d is *constructible* if it is a simplex, or if it can be written as $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$ are constructible of dimensions d, d, and $d-1$ respectively.

Hachimori [Hac00] investigated the question of whether constructibility implies partitionability. Our counterexample to the Partitionability Conjecture is in fact constructible, resolving this question as well.

3. The counterexample

We first give a general construction that reduces the problem of finding a counterexample to the problem of constructing a certain kind of non-partitionable Cohen–Macaulay relative complex.

Theorem 3.1. *Let $Q = (X, A)$ be a relative complex such that*

(i) *X and A are Cohen–Macaulay;*
(ii) *A is an induced subcomplex of X of codimension at most 1; and*
(iii) *Q is not partitionable.*

Let k be the total number of faces of A, let $N > k$, and let $C = C_N$ be the simplicial complex constructed from N disjoint copies of X identified along the subcomplex A. Then C is Cohen–Macaulay and not partitionable.

Proof. First, C is Cohen–Macaulay by Proposition 2.3. Second, suppose that C has a partitioning $\mathcal{P}$. Let $X_1, X_2, \ldots, X_N$ be the N copies of X. By the pigeonhole principle, since $N > k$, there is some copy of X, say X_N, none of whose facets is matched to a face in A. Let $[R_1, F_1], \ldots, [R_\ell, F_\ell]$ be the intervals in $\mathcal{P}$ for which $F_i \in X_N$; then

$$\bigcup_{i=1}^{\ell} [R_i, F_i] \quad \subseteq \quad X_N \setminus A. \tag{4}$$

No other interval in $\mathcal{P}$ can intersect $X_N \setminus A$ nontrivially, so in fact equality must hold in (4). But then (4) is in fact a partitioning of $X_N \setminus A = Q$, which was assumed to be non-partitionable. □

Remark 3.2. It is easy to see that a subcomplex $A \subset X$ is an induced subcomplex if and only if every minimal face of $X \setminus A$ has dimension 0. Therefore, this condition may be viewed as a restriction on the relative complex (X, A).

3.1. **The construction.** Throughout, we abbreviate the simplex on vertices $\{v_1, \ldots, v_k\}$ by $v_1 \cdots v_k$. Our construction begins with Ziegler's nonshellable 3-ball Z, which is a nonshellable triangulation of the 3-ball with 10 vertices labeled $0, 1, \ldots, 9$, and the following 21 facets [Zie98, §4]:

$$\begin{array}{lllllll} 0123, & 0125, & 0237, & 0256, & 0267, & 1234, & 1249, \\ 1256, & 1269, & 1347, & 1457, & 1458, & 1489, & 1569, \\ 1589, & 2348, & 2367, & 2368, & 3478, & 3678, & 4578. \end{array}$$

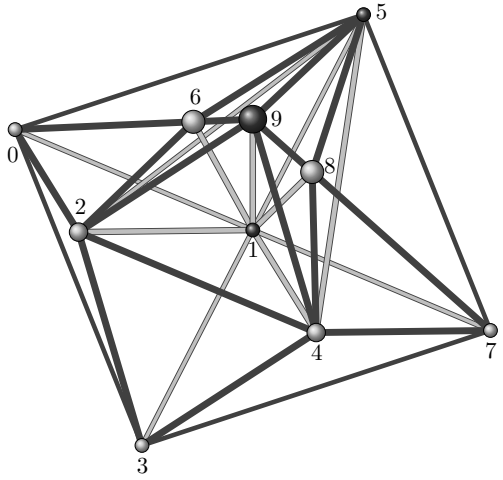

FIGURE 1. A perspective view of the 1-skeleton of $\bar{Q}$ from the top. Dark edges are exterior edges visible from the top; light edges are interior edges, or exterior edges at the bottom. Light vertices are in A.

The complex Z is not shellable, but it is partitionable, Cohen–Macaulay and, in fact, constructible [Hac01].

Let B be the induced subcomplex $Z|_{\{0,2,3,4,6,7,8\}}$. That is, B is the pure 3-dimensional complex with facets

$$0237, \quad 0267, \quad 2367, \quad 2368, \quad 2348, \quad 3678, \quad 3478.$$

The given order is a shelling of B; in particular B is Cohen–Macaulay. Define Q to be the relative complex $Q = (Z, B)$. Then Q is also Cohen–Macaulay by [Duv96, Corollary 3.2].

The facets of Q are

$$\begin{array}{llllllll} 1249, & 1269, & 1569, & 1589, & 1489, & 1458, & 1457, \\ 4578, & 1256, & 0125, & 0256, & 0123, & 1234, & 1347. \end{array} \tag{5}$$

The minimal faces of Q are just the vertices 1, 5, 9. We can picture Q easily by considering its combinatorial closure $\bar{Q}$, that is, the 3-dimensional simplicial complex generated by the facets (5). In fact $\bar{Q}$ is a shellable ball; the ordering of facets given in (5) is a shelling. The complement $A = \bar{Q} \setminus Q = \bar{Q}|_{\{0,2,3,4,6,7,8\}}$ is the shellable 2-ball with facets

$$026, \quad 023, \quad 234, \quad 347, \quad 478. \tag{6}$$

Thus $Q = (\bar{Q}, A)$. The f- and h-vectors of these complexes are

$$\begin{aligned} f(\bar{Q}) &= (1, 10, 31, 36, 14), & h(\bar{Q}) &= (1, 6, 7, 0, 0), \\ f(A) &= (1, 7, 11, 5, 0), & h(A) &= (1, 4, 0, 0, 0), \\ f(Q) &= (0, 3, 20, 31, 14), & h(Q) &= (0, 3, 11, 0, 0). \end{aligned}$$

The 1-skeleton of $\bar{Q}$ is shown in Figure 1. The triangles on the boundary of Q, i.e., those contained in exactly one facet, are illustrated in Figure 2, which shows

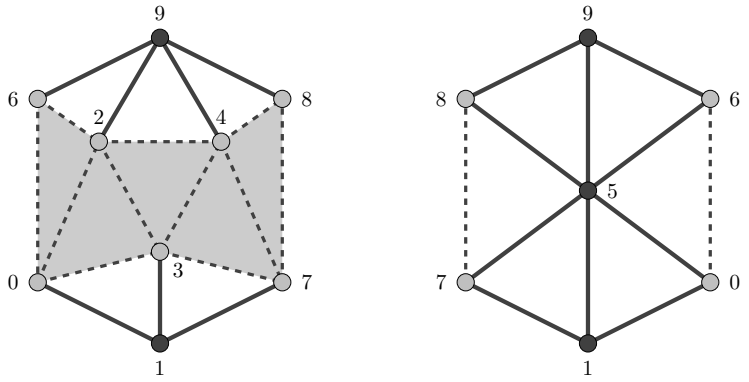

FIGURE 2. *Left:* A front view of Q. *Right:* A back view of Q. The shaded and dashed faces are in A.

the boundary of Q as seen from the front (left) and back (right). The five shaded triangles are the facets of A, and hence are missing from Q.

In what follows, we will use the fact that the triple transposition $\tau = (0\ 7)(2\ 4)(6\ 8)$ is a simplicial automorphism of $\bar{Q}$. This symmetry is apparent as a reflection through the plane containing vertices 1, 3, 5, and 9 in Figure 1, and as a vertical reflection in each part of Figure 2.

Theorem 3.3. *The relative complex Q is not partitionable.*

Proof. Suppose that Q admits a partitioning $\mathcal{P}$. We will show that a particular minimal face, namely vertex 5, must simultaneously belong to two intervals of the partitioning, which is a contradiction.

For each facet $F \in Q$, denote by $I_F = [R_F, F]$ the interval of $\mathcal{P}$ with top element F.

For each triangle T on the boundary, there is only one interval that can contain T. In particular, $489 \in I_{1489}$. It follows that $148 \notin I_{1489}$, for otherwise $148 \cap 489 = 48 \in I_{1489}$, but $48 \notin Q$. Therefore $148 \in I_{1458}$, since 1458 is the only other facet containing 148. Then $458 \notin I_{1458}$, again because $148 \cap 458 = 48 \notin Q$, and thus $45 \notin I_{1458}$. The other two facets that contain 45 are 4578 and 1457. Therefore, either $45 \in I_{4578}$ or $45 \in I_{1457}$. On the other hand, these are also the only two facets that contain the edge 57. Since

$$45, 57 \ \subset \ 457 \ \subset \ 1457, 4578$$

the edges 45 and 57 must belong to the same interval of $\mathcal{P}$ (namely, whichever one of I_{1457}, I_{4578} contains 457). But then that interval must also contain $45 \cap 57 = 5$. We have shown that

$$\text{either } 5 \in I_{1457} \text{ or } 5 \in I_{4578}. \tag{7}$$

By applying the automorphism τ to the above argument, we conclude that

$$\text{either } 5 \in I_{0125} \text{ or } 5 \in I_{0256}. \tag{8}$$

But (7) and (8) cannot both be true, and we have reached a contradiction. □

We can now give an explicit description of our counterexample to the Partitionability Conjecture. Since $X = \bar{Q}$ and A are both shellable balls, they are

Cohen–Macaulay. We may therefore apply Theorem 3.1, with $N = 25$ (since A has 24 faces total).

Theorem 3.4. *Let $X = \bar{Q}$ be the combinatorial closure of Q, and let $A = X \setminus Q$. That is, X and A are the absolute simplicial complexes whose facets are listed in* (5) *and* (6)*, respectively. Then the simplicial complex C_{25} constructed in Theorem 3.1 is Cohen–Macaulay and non-partitionable.*

The f-vector is $f(C_{25}) = f(A) + 25f(Q) = (1, 82, 511, 780, 350)$.

For this particular construction, the full power of Theorem 3.1 is not necessary; there is a much smaller counterexample.

Theorem 3.5. *Let Q, A, and $X = \bar{Q}$ be as described above. Then the simplicial complex C_3 obtained by gluing together three copies of X along A is Cohen–Macaulay and non-partitionable.*

Proof. Suppose that C_3 is partitionable. By the pigeonhole principle, at least one of the three copies of Q inside C_3 has no facets matched to either edge 48 or its image under τ, edge 26. These two edges are the only two faces of A that occur in the argument of Theorem 3.3. Therefore, that argument applies to this copy of Q, and once again we conclude that (7) and (8) must both hold, a contradiction. □

The f-vector is $f(C_3) = f(A) + 3f(Q) = (1, 16, 71, 98, 42)$. We do not know if there exists a smaller counterexample (for example, the complex C_2 obtained by gluing two copies of X together along A is partitionable). In particular, it is still open whether every *two*-dimensional Cohen–Macaulay simplicial complex is partitionable; see Hachimori [Hac08].

We have previously observed that X and A are shellable. We note that X and A are contractible, and it is easily seen that X deformation-retracts onto A, so C_3 is contractible as well, although it is not homeomorphic to a ball.

Remark 3.6. There is a much smaller *relative* simplicial complex that is Cohen–Macaulay but not partitionable, with f-vector $(0, 0, 5, 10, 5)$. This complex can be written as $Q' = (X', A')$, where $X' = \overline{Q'} = Z|_{\{1,4,5,7,8,9\}}$ is the complex with facets

$$1589,\ 1489,\ 1458,\ 1457,\ 4578,$$

and A' is the (non-induced) subcomplex of X' with facets

$$489,\ 589,\ 578,\ 157.$$

These complexes are shellable balls of dimensions 3 and 2 respectively (the given orders of facets are shelling orders), and A' is contained in the boundary of X' (note that each facet in A' is contained in only one facet of X'), so Q' is Cohen–Macaulay by [Sta87, Corollary 5.4]. On the other hand, one can check directly that there is no partitioning of Q'. Because A' is not an induced subcomplex, it is not possible to obtain a counterexample to the Partitionability Conjecture by applying Theorem 3.1.

Remark 3.7. It is easily seen that C_3 is constructible. Therefore, it furnishes a counterexample not only to the Partitionability Conjecture, but also to the conjecture that every constructible simplicial complex is partitionable [Hac00, §4]. Furthermore, since all constructible complexes are Cohen–Macaulay [BH93, p. 219], the constructibility and non-partitionability of C_3 are sufficient to disprove the Partitionability Conjecture.

3.2. **Stanley depth.** Let $\Bbbk$ be a field and $S = \Bbbk[x_1, \ldots, x_n]$, and let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* $\mathcal{D}$ of M is a vector space decomposition

$$M = \bigoplus_{i=1}^{r} \Bbbk[X_i] \cdot m_i$$

where each X_i is a subset of $\{x_1, \ldots, x_n\}$ and each m_i is a homogeneous element of M. The *Stanley depth* of M is defined as

$$\operatorname{sdepth} M = \max_{\mathcal{D}} \{\min(|X_1|, \ldots, |X_r|)\},$$

where $\mathcal{D}$ ranges over all Stanley decompositions of M. If Φ is an (absolute or relative) simplicial complex, then we define its Stanley depth to be the Stanley depth of its associated Stanley–Reisner ring or module. This invariant has received substantial recent attention [PSFTY09, Her13], centering on the Depth Conjecture of Stanley [Sta82, Conjecture 5.1], which we now restate.

Conjecture 1.2 (Depth Conjecture)**.** *For all $\mathbb{Z}^n$-graded S-modules M,*

$$\operatorname{sdepth} M \geq \operatorname{depth} M.$$

Herzog, Jahan and Yassemi [HJY08, Corollary 4.5] proved that if Δ is a Cohen–Macaulay simplicial complex whose Stanley–Reisner ring [Sta96, §II.1] is $\Bbbk[\Delta] := S/I_\Delta$ (so that $\operatorname{depth} \Bbbk[\Delta] = \dim \Bbbk[\Delta] = \dim \Delta + 1$), then Conjecture 1.2 holds for $\Bbbk[\Delta]$ if and only if Δ is partitionable. Therefore, our construction provides a counterexample to the Depth Conjecture. Katthän has conjectured that the inequality $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I - 1$ holds for every monomial ideal I; for a detailed exposition and the evidence for this conjecture, see [Kat16].

A smaller counterexample to Conjecture 1.2 is provided by the relative complex Q' in Remark 3.6. The depth of each of C_3 and Q' is easily seen to be 4, but the Stanley depth of each of C_3 and Q' is 3. The Stanley depth computations were made by Katthän [Kat], using the algorithm developed by Ichim and Zarojanu [IZ14].

4. Open questions

Now that we know that Cohen–Macaulayness and even constructibility are not sufficient to guarantee partitionability, it is natural to ask what other conditions do suffice. Hachimori defined a related but more restricted class of *strongly constructible* complexes and showed that they are always partitionable [Hac00, Corollary 4.7]. Here are two additional possibilities, inspired by what our counterexample C_3 is *not*. First, C_3 is not homeomorphic to a ball, because the triangles in A are each contained in three facets. On the other hand, balls are Cohen–Macaulay, motivating the following question:

Question 4.1. *Is every simplicial ball partitionable?*

This conjecture is true if we further assume the ball is convexly realizable, by [Sta96, Proposition III.2.8]; see also [KS91]. On the other hand, there exist non-convex simplicial balls in dimensions as small as 3; see, e.g., [Lut04b, Lut08].

Garsia [Gar80, Remark 5.2] proposed the Partitionability Conjecture for the special class of order complexes of *Cohen–Macaulay posets* (see also [Bac76, Bac80, BGS82]), which give rise to balanced Cohen–Macaulay simplicial complexes. Recall that a d-dimensional simplicial complex is *balanced* if its vertices can be colored with

$d+1$ colors so that every facet has one vertex of each color. For instance, if P is a ranked poset, then its order complex is easily seen to be balanced by associating colors with ranks. The complex $\bar{Q}$ with facets listed in (5) is not balanced (because its 1-skeleton is not 4-colorable), hence neither is C_3 or C_{25}, nor indeed C_N for any N.

Question 4.2. *Is every balanced Cohen–Macaulay simplicial complex partitionable?*

Although Cohen–Macaulay complexes are not necessarily partitionable, their h-vectors are still nice; they are always non-negative and in fact coincide with the h-vectors of shellable complexes. Without the Partitionability Conjecture, the question remains:

Question 4.3. *What does the h-vector of a Cohen–Macaulay simplicial complex count?*

One answer is given by [DZ01], where it is shown that every simplicial complex can be decomposed into *Boolean trees* indexed by iterated Betti numbers; see [DZ01, Corollary 3.5]. The starting point of that paper is a conjecture of Kalai [Kal02, Conjecture 22] that any simplicial complex can be partitioned into intervals in a way related to algebraic shifting. Kalai's conjecture would have implied that simplicial complexes could be decomposed into Boolean intervals. Such a decomposition into intervals, however, would have implied the Partitionability Conjecture. Hence our result provides a counterexample to Kalai's conjecture. Moreover, the decomposition in [DZ01] may be best possible at this level of generality.

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