# GEOMETRY OF GRAPH VARIETIES

JEREMY L. MARTIN

Abstract. A *picture* $\mathbf{P}$ of a graph $G = (V, E)$ consists of a point $\mathbf{P}(v)$ for each vertex $v \in V$ and a line $\mathbf{P}(e)$ for each edge $e \in E$, all lying in the projective plane over a field $\mathbf{k}$ and subject to containment conditions corresponding to incidence in $G$. A *graph variety* is an algebraic set whose points parametrize pictures of $G$. We consider three kinds of graph varieties: the *picture space* $\mathcal{X}(G)$ of all pictures; the *picture variety* $\mathcal{V}(G)$, an irreducible component of $\mathcal{X}(G)$ of dimension $2|V|$, defined as the closure of the set of pictures on which all the $\mathbf{P}(v)$ are distinct; and the *slope variety* $\mathcal{S}(G)$, obtained by forgetting all data except the slopes of the lines $\mathbf{P}(e)$. We use combinatorial techniques (in particular, the theory of *combinatorial rigidity*) to provide substantial geometric information on these varieties. We obtain the following results:

1. A new proof of the equality of two matroids studied in rigidity theory.
2. A description and combinatorial interpretation of equations defining each variety set-theoretically.
3. A description of the irreducible components of $\mathcal{X}(G)$.
4. A proof that $\mathcal{V}(G)$ and $\mathcal{S}(G)$ are Cohen-Macaulay when $G$ satisfies a sparsity condition, *rigidity independence*.

## 1. Introduction

This paper initiates the study of certain algebraic varieties that parametrize plane pictures $\mathbf{P}$ of a given graph $G$, with vertices $v$ and edges $e$ represented respectively by points $\mathbf{P}(v) \in \mathbb{P}^2$ and lines $\mathbf{P}(e)$ connecting them in pairs. Three such varieties naturally arise. First of all, there is the *picture space* $\mathcal{X}(G)$ of all pictures of $G$. Usually, $\mathcal{X}(G)$ is not irreducible. It is therefore natural to restrict attention to a second variety, namely the irreducible component of $\mathcal{X}(G)$ containing as a dense set those pictures in which the points $\mathbf{P}(v)$ are all distinct. This most generic component of the picture space is called the *picture variety* $\mathcal{V}(G)$. As we shall see, $\mathcal{V}(G)$ is cut out in $\mathcal{X}(G)$ purely by equations relating the slopes of the lines $\mathbf{P}(e)$. The crucial matter for the whole study is to understand the relations among these slopes. This leads us to consider the *slope variety* $\mathcal{S}(G)$, which is essentially the projection of $\mathcal{V}(G)$ on coordinates $m_e$ giving the slopes of the lines $\mathbf{P}(e)$.

In a sequel to this paper, we study intensively the case where $G$ is the complete graph $K_n$. There we will obtain very precise results, including the proof for $K_n$ of some conjectures mentioned below, along with remarkable connections to the combinatorics of matchings and planar trees. Note that the problem of describing the slope variety $\mathcal{S}(K_n)$ is of a very classical kind: it is exactly the problem of determining all relations among the slopes of the $\binom{n}{2}$ lines connecting $n$ general points in the plane.

Here we consider the features of varieties associated with an arbitrary graph $G = (V, E)$. We shall see that the *generic rigidity matroid* $\mathcal{M}(V)$ studied by Laman *et. al.* [5], [3] makes an appearance here as the algebraic dependence matroid of the slopes. For each set of edges forming a circuit in the matroid $\mathcal{M}(V)$, we can write down an explicit determinantal formula for the essentially unique polynomial relation among the corresponding slopes $m_e$. We prove that precisely these relations cut out $\mathcal{V}(G)$ in $\mathcal{X}(G)$ set-theoretically. We also show how the full component structure of $\mathcal{X}(G)$ can be economically described in terms of the rigidity matroid, and show that when $\mathcal{X}(G) = \mathcal{V}(G)$, this variety has Cohen-Macaulay singularities.

The slope relation induced by each circuit in $\mathcal{M}(V)$ turns out to be a very remarkable polynomial. All its terms are squarefree, and they have a surprising combinatorial interpretation in terms of decompositions of the given circuit into complementary spanning trees. We conjecture that the slope relations should cut out $\mathcal{V}(G)$ scheme-theoretically as well as set-theoretically. We further suspect that they may always form a *universal Gröbner basis* for the ideal of the slope variety, and moreover, that both $\mathcal{S}(G)$ and $\mathcal{V}(G)$ are always Cohen-Macaulay.

When we first embarked upon the study of graph varieties, before obtaining the results indicated above, we already had some reasons to think they might be interesting. Since these reasons remain relevant, let us mention them briefly. Graph varieties are a particular class of *configuration varieties*—subvarieties in a product of Grassmannians defined by containment conditions among various subspaces of a fixed space. Other examples of configuration varieties are Bott-Samelson-Demazure varieties and somewhat more general varieties introduced and studied by Magyar [6]. The latter have very special geometric properties, and it is natural to inquire to what extent these are shared by more general configuration varieties. Graph varieties provide the simplest non-trivial examples not fitting into Magyar's framework. Furthermore, for $G = K_n$, the picture variety $\mathcal{V}(G)$ is a blowdown of the Fulton-Macpherson *compactification of configuration space* [2], which desingularizes it. For general $G$, the same relation holds between $\mathcal{V}(G)$ and the DeConcini-Procesi *wonderful model of subspace arrangements* [1]. We expect that $\mathcal{V}(G)$ should not only be Cohen-Macaulay but should have rational singularities. This would be equivalent to a cohomology vanishing theorem for certain line bundles on the wonderful model, raising an important question for further study.

This paper is essentially Chapter 1 of the author's Ph.D. dissertation [7], written under the supervision of Mark Haiman. The author wishes to thank Prof. Haiman for his advice and encouragement.

## 2. Definitions

We work over an algebraically closed field $\mathbf{k}$. Affine and projective $n$-space over $\mathbf{k}$ are denoted by $\mathbb{A}^n$ and $\mathbb{P}^n$ respectively. The Grassmannian variety of all $r$-dimensional vector subspaces of $\mathbf{k}^n$ is denoted by $\mathrm{Gr}(n, r)$.

A *graph* $G$ is a pair $(V, E)$, where $V = V(G)$ is a finite set of *vertices* and $E = E(G)$ is a set of *edges*, or unordered pairs of distinct vertices $\{v, w\}$. We frequently abbreviate $\{v, w\}$ by $vw$ when no confusion can arise (for instance, when the vertices are one-digit positive integers). The vertices $v, w$ are called the *endpoints* of the edge $vw$. A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ with $V' \subset V$ and $E' \subset E$.

For $V' \subset V$, we define

$$\begin{aligned} K(V') &= \{vw \mid v, w \in V',\ v \neq w\}, \\ E(V') &= E \cap K(V'). \end{aligned}$$

The *complete graph on* $V$ is the graph $(V, K(V))$. We write $K_n$ for the complete graph on $\{1, \ldots, n\}$.

For $E' \subset E$ and $v \in V$, we define

$$\begin{aligned} \mathrm{val}_{E'}(v) &= \bigl|\{e \in E' \mid v \in e\}\bigr| \qquad \text{and} \\ V(E') &= \{v \in V \mid \mathrm{val}_{E'}(v) > 0\}. \end{aligned}$$

For $v_1, \ldots, v_s \in V$, we define

$$(v_1, \ldots, v_s) = \{v_1v_2,\ v_2v_3,\ \ldots,\ v_{s-1}v_s\} \subset E.$$

If the $v_i$ are all distinct, then $(v_1, \ldots, v_s)$ is called a *path*. If $v_1, \ldots, v_{s-1}$ are distinct and $v_1 = v_s$, then $(v_1, \ldots, v_s)$ is called a *polygon* or $(s-1)$*-gon*. A polygon is more usually called a "cycle" or "circuit," but we wish to reserve these words for other uses.

A graph $G = (V, E)$ is *connected* if every pair of vertices are joined by a path, and is a *forest* if at most one such path exists for every pair. A connected forest is called a *tree*. A *spanning tree* of $G$ (or of $V$) is a tree $T \subset E$ with $V(T) = V$. A *connected component* of $G$ is a maximal connected subgraph; every graph has a unique decomposition into connected components (where some components may be isolated vertices).

A *partition* of a finite set $V$ is a set $\mathcal{A} = \{A_1, \ldots, A_s\}$ of pairwise disjoint subsets of $V$ whose union is $V$. The sets $A_i$ are called the *blocks* of $\mathcal{A}$. We write $\sim_{\mathcal{A}}$ for the equivalence relation on $V$ whose equivalence classes are the blocks of $\mathcal{A}$. We distinguish two extreme cases: the *discrete partition* $\mathcal{D}_V$, all of whose blocks are singletons, and the *indiscrete partition* $\mathcal{I}_V$, which has only one block. Finally, if $\mathcal{A}$ and $\mathcal{B}$ are partitions of $V$, then we say that $\mathcal{A}$ *refines* $\mathcal{B}$, written $\mathcal{A} \preceq \mathcal{B}$, if every block of $\mathcal{A}$ is contained in some block of $\mathcal{B}$. It is elementary that refinement is a partial ordering.

## 3. The Picture Space and Picture Variety of a Graph

Throughout this section, we consider a graph $G = (V, E)$ with $|V| = n$ and $|E| = r$.

Define

$$\mathrm{Gr}(G) = \left(\prod_{v \in V} \mathrm{Gr}(3,1)\right) \times \left(\prod_{e \in E} \mathrm{Gr}(3,2)\right). \tag{1}$$

Note that $\mathrm{Gr}(3,1)$ is just the projective plane $\mathbb{P}^2$, and $\mathrm{Gr}(3,2)$ may be identified with the set of lines in $\mathbb{P}^2$.

For $\mathbf{P} \in \mathrm{Gr}(G)$, $v \in V$, and $e \in E$, we write $\mathbf{P}(v)$ and $\mathbf{P}(e)$ for the projections of $\mathbf{P}$ on the indicated factors in (1).

**Definition 3.1.** A *picture* of $G$ is a point $\mathbf{P} \in \mathrm{Gr}(G)$ such that

$$v \in e \implies \mathbf{P}(v) \in \mathbf{P}(e). \tag{2}$$

The *picture space* $\mathcal{X}(G)$ is the set of all pictures of $G$.

Note that $\mathcal{X}(G)$ is Zariski closed in $\mathrm{Gr}(G)$, since the conditions (2) may be expressed in terms of the Plücker coordinates. Note also that if $G_1, \ldots, G_s$ are the connected components of $G$, then

$$\mathcal{X}(G) \cong \mathcal{X}(G_1) \times \ldots \times \mathcal{X}(G_s). \tag{3}$$

The equations defining $\mathcal{X}(G)$ in homogeneous coordinates are awkward to work with explicitly. However, all the geometric information we will require about $\mathcal{X}(G)$ can be recovered from the following affine open subset of it, on which the defining equations assume a more manageable form.

**Definition 3.2.** Fix homogeneous coordinates $[a_0 : a_1 : a_2]$ on $\mathbb{P}^2$, identifying $\mathbb{A}^2$ with the points for which $a_0 \neq 0$ and giving $x = a_1/a_0$, $y = a_2/a_0$ as affine coordinates on $\mathbb{A}^2$. The *affine picture space* $\tilde{\mathcal{X}}(G)$ is the open subvariety of $\mathcal{X}(G)$ consisting of pictures $\mathbf{P}$ such that all points $\mathbf{P}(v)$ lie in $\mathbb{A}^2$ and no line $\mathbf{P}(e)$ is parallel to the $y$-axis.

Note that $\tilde{\mathcal{X}}(G)$ is open and dense in $\mathcal{X}(G)$, and that $\mathcal{X}(G)$ is covered by finitely many copies of $\tilde{\mathcal{X}}(G)$. In addition $\tilde{\mathcal{X}}(G)$ has affine coordinates

$$\begin{array}{rl} x_v, y_v : & v \in V, \\ m_e, b_e : & e \in E, \end{array} \tag{4}$$

where $m_e$ and $b_e$ denote respectively the slope and $y$-intercept of the line $\mathbf{P}(e)$. Thus $\tilde{\mathcal{X}}(G)$ is the vanishing locus (in $\mathbb{A}^{2n+2r}$, identified with an open subset of $\mathrm{Gr}(G)$) of the ideal generated by the $2r$ equations

$$\begin{array}{c} y_v = m_e x_v + b_e, \\ y_w = m_e x_w + b_e, \end{array} \tag{5}$$

for each edge $e = vw$. Eliminating the variables $b_e$ from (5) produces $r$ equations

$$(y_v - y_w) = m_e(x_v - x_w). \tag{6}$$

We may also eliminate the variables $y_v$. For each polygon $P = (v_1, \ldots, v_s, v_1)$ of $G$, we sum the equations (6) over the edges of $P$, obtaining the equation

$$L(P) = 0, \tag{7}$$

where

$$L(P) = \sum_{i=1}^{s} m_{e_i}(x_{v_i} - x_{v_{i+1}}). \tag{8}$$

where $e_i = v_i v_{i+1}$ and the indices are taken modulo $s$. Given a solution $(\mathbf{m}, \mathbf{x})$ of the equations (7), we may choose one $y$-coordinate arbitrarily and use (5) and (6) to recover the coordinates $y_v$ and $b_e$. Putting

$$\begin{array}{rcll} R_G &=& \mathbf{k}[m_e & \mid\ e \in E], \\ R'_G &=& \mathbf{k}[m_e, x_v & \mid\ e \in E,\ v \in V], \end{array} \tag{9}$$

we see that $\tilde{\mathcal{X}}(G) \cong \mathbb{A}^1 \times X$, where $X$ is the subscheme of $\operatorname{Spec} R'_G \cong \mathbb{A}^{|V|+|E|}$ defined set-theoretically by the equations (7).

There is a natural decomposition of $\mathcal{X}(G)$ into locally closed irreducible nonsingular subvarieties, which we call *cellules*. The decomposition is somewhat analogous to the decomposition of a flag variety into Schubert cells.

**Definition 3.3.** Let $\mathcal{A} = \{A_1, \ldots, A_s\}$ be a partition of $V$. The *cellule* of $\mathcal{A}$ in $\mathcal{X}(G)$ is the quasiprojective subvariety

$$\mathcal{X}_{\mathcal{A}}(G) = \{\mathbf{P} \in \mathcal{X}(G) \ \mid\ \mathbf{P}(v) = \mathbf{P}(w) \iff v \sim_{\mathcal{A}} w\}. \tag{10}$$

Unlike a Schubert cell, a cellule $\mathcal{X}_{\mathcal{A}}(G)$ is not isomorphic to an affine space. It is, however, a smooth fiber bundle. To see this, let $\mathbf{P} \in \mathcal{X}_{\mathcal{A}}(G)$ and $e = vw \in E$. If $v \sim_{\mathcal{A}} w$, then the set of lines in $\mathbb{P}^2$ through $\mathbf{P}(v) = \mathbf{P}(w)$ is isomorphic to $\mathbb{P}^1$, and $\mathbf{P}(e)$ may take any value in that set. If on the other hand $v \not\sim_{\mathcal{A}} w$, then $\mathbf{P}(e)$ is determined uniquely by $\mathbf{P}(v)$ and $\mathbf{P}(w)$. Therefore, putting

$$q = \big|\{vw \in E \ \mid\ v \sim_A w\}\big| \tag{11}$$

and

$$U = \big\{(p_1, \ldots, p_s) \in (\mathbb{P}^2)^s \ \mid\ i \neq j \implies p_i \neq p_j\big\}, \tag{12}$$

we see that $\mathcal{X}_{\mathcal{A}}(G)$ has the bundle structure

$$\begin{array}{ccc} (\mathbb{P}^1)^q & \to & \mathcal{X}_A(G) \\ & & \downarrow \\ & & U \end{array} \tag{13}$$

and in particular

$$\dim \mathcal{X}_{\mathcal{A}}(G) = 2s + \big|\{vw \in E \ \mid\ v \sim_{\mathcal{A}} w\}\big|. \tag{14}$$

**Definition 3.4.** Let $G = (V, E)$. A picture $\mathbf{P} \in \mathcal{X}(G)$ is called *generic* if no two of the points $\mathbf{P}(v)$ coincide. The *discrete cellule* $\mathcal{V}^\circ(G)$ is defined as the set of all generic pictures of $G$. Note that $\mathcal{V}^\circ(G) = \mathcal{X}_{\mathcal{D}}(G)$, where $\mathcal{D} = \mathcal{D}_V$ is the discrete partition of $V$ (the partition into singleton sets). The *picture variety* of $G$ is defined as

$$\mathcal{V}(G) = \overline{\mathcal{V}^\circ(G)}.$$

This is an irreducible component of $\mathcal{X}(G)$. By (14), we have

$$\dim \mathcal{V}(G) = \dim \mathcal{V}^\circ(G) = 2|V(G)|. \tag{15}$$

q The *affine picture variety* of $G$ is defined as

$$\tilde{\mathcal{V}}(G) = \mathcal{V}(G) \cap \tilde{\mathcal{X}}(G).$$

**Remark 3.5.** For $G = (V, E)$ and $W \subset V$, the *coincidence locus* of $W$ is defined as

$$C_W = C_W(G) = \{\mathbf{P} \in \mathcal{X}(G) \ \mid \ \mathbf{P}(v) = \mathbf{P}(w) \text{ for all } v, w \in W\}. \tag{16}$$

Let $G_0$ be the graph with vertices $V = V(G)$ and no edges. We may regard $\mathcal{V}(G)$ as the simultaneous blowup of $(\mathbb{P}^2)^n = \mathcal{X}(G_0)$ along the coincidence loci $C_e$ for all $e = vw \in E$. Indeed, the further blowup of $(\mathbb{P}^2)^n$ along all $C_W$, where $W \subset V$ is connected, is an instance of the "wonderful model of subspace arrangements" of DeConcini and Procesi [1]. This blowup is a desingularization of $\mathcal{V}(G)$. When $G$ is the complete graph $K_n$, this is the "compactification of configuration space" of Fulton and MacPherson [2].

Note that the only cellule which is closed in $\mathcal{X}(G)$ is the *indiscrete cellule* $\mathcal{X}_{\mathcal{I}}(G)$, where $\mathcal{I} = \mathcal{I}_V$ is the indiscrete partition of $V$ (the partition with just one block).

**Example 3.6.** Let $G = K_2$. Denote by $\mathcal{D}$ and $\mathcal{I}$ respectively the discrete and indiscrete partitions of $V = V(G) = \{1, 2\}$. The picture space $\mathcal{X}(K_2)$ is the blowup of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal

$$\Delta = \{(p_1, p_2) \in \mathbb{P}^2 \times \mathbb{P}^2 \ \mid \ p_1 = p_2\}.$$

The blowup map

$$\pi : \ \mathcal{X}(K_2) \to \mathbb{P}^2 \times \mathbb{P}^2$$

is just the projection on the vertex coordinates. The exceptional divisor $\pi^{-1}(\Delta)$ is the indiscrete cellule, which has dimension 3. Since there are no partitions of $V$ other than $\mathcal{D}$ and $\mathcal{I}$, the complement of $\mathcal{X}_{\mathcal{I}}(K_2)$ is $\mathcal{V}^\circ(K_2)$, which has dimension 4 and is dense in $\mathcal{X}(K_2)$. Thus $\mathcal{V}(K_2) = \mathcal{X}(K_2)$.

**Example 3.7.** In general, the picture space $\mathcal{X}(G)$ is not irreducible. The first example, and in many ways the fundamental one, is the graph $K_4$. Denote by $\mathcal{D}$ and $\mathcal{I}$ respectively the discrete and indiscrete partitions of $\{1, 2, 3, 4\}$. By (14), we have

$$\dim \mathcal{V}^\circ(K_4) = 8 = \dim \mathcal{X}_{\mathcal{I}}(K_4),$$

so $\mathcal{X}_{\mathcal{I}}(K_4)$ is too big to be contained in the closure of $\mathcal{V}^\circ(K_4)$. Hence $\mathcal{V}(K_4) \neq \mathcal{X}(K_4)$. We will subsequently show that the irreducible components of $\mathcal{X}(K_4)$ are precisely $\mathcal{V}(K_4)$ and $\mathcal{X}_{\mathcal{I}}(K_4)$.

We will soon see that the polynomials defining $\tilde{\mathcal{V}}(G)$ as a subvariety of $\tilde{\mathcal{X}}(G)$ involve only the variables $m_e$. In order to study these polynomials in isolation, we define a third type of graph variety. As before, we identify $\mathbb{A}^2$ with an open affine subset of $\mathbb{P}^2$.

**Definition 3.8.** Let $U$ be the (dense) set of pictures $\mathbf{P} \in \mathcal{V}(G)$ such that no $\mathbf{P}(e)$ is the line at infinity. Accordingly, for each $e$, $\mathbf{P}(e) \cap \mathbb{A}^2$ is an affine line of the form

$$\big\{(x, y) \ \mid \ \alpha_e x + \beta_e y = 1,$$

with a well-defined "slope" $[\alpha_e : \beta_e] \in \mathbb{P}^1$. Forgetting all the data of $\mathbf{P}$ except the slopes gives a map

$$\phi : U \to (\mathbb{P}^1)^r. \tag{17}$$

We define the *slope variety* $\mathcal{S}(G)$ as the image of $\phi$. An element of $\mathcal{S}(G)$ is called a *slope picture* of $G$. If $a_1^e \neq 0$ for all $e$, then we have an *affine slope picture*. Setting $m_e = a_0^e / a_1^e$, we may regard an affine slope picture as a point $\mathbf{m} = (m_e \ \mid \ e \in E)$ of $\operatorname{Spec} R_G$. The algebraic set $\tilde{\mathcal{S}}(G)$ of all affine slope pictures is called the *affine* *slope variety* of $G$. $\tilde{\mathcal{S}}(G)$ may also be defined as the projection of $\tilde{\mathcal{V}}(G)$ on the coordinates $m_e$; since $\tilde{\mathcal{V}}(G)$ is irreducible, so is $\tilde{\mathcal{S}}(G)$.

**Remark 3.9.** Restricting $\phi$ to $\tilde{\mathcal{V}}(G)$ produces a surjective map

$$\phi : \tilde{\mathcal{V}}(G) \twoheadrightarrow \tilde{\mathcal{S}}(G). \tag{18}$$

Note that every fiber of $\phi$ has dimension at least 3, because translation and scaling do not affect slopes of lines.

We will show that the same ideal of $R_G$ cuts out $\tilde{\mathcal{S}}(G)$ set-theoretically as a subvariety of $\operatorname{Spec} R_G$, and $\tilde{\mathcal{V}}(G)$ as a subvariety of $\tilde{\mathcal{X}}(G)$. To study this ideal, we will use tools from the theory of combinatorial rigidity.

## 4. Combinatorial Rigidity Theory

The behavior of graph varieties is governed in various ways by a certain combinatorial object, the *generic rigidity matroid*. Accordingly, we begin this section by sketching the elements of rigidity theory, collecting several facts which we will need later. (Our treatment here is necessarily brief; for a detailed exposition, see [3] or [9].) The main new result of this section, Theorem 4.5, describes the fundamental connection between the purely combinatorial theory of rigidity and the geometry of graph varieties.

Let $G = (V, E)$ be a connected graph, and $\mathbf{P}$ a generic picture of $G$ defined over $\mathbb{R}$. For the sake of easy visualization, we may take $k = \mathbb{R}$ for the moment (the requirement that $k$ be algebraically closed is not needed for the notion of rigidity). Imagine a physical model of $\mathbf{P}$ in which the vertices and edges are represented by "ball joints" and "rods" respectively. The rods are considered to be fixed in length, but are free to rotate about the joints in the plane of the picture. Intuitively, $G$ is *length-rigid*, or simply *rigid*, if the physical realization of any generic picture of $G$ will "hold its shape." More precisely, $G$ is rigid if the distance between any two vertices in a generic picture is determined by the lengths of the edges in $E$, up to finitely many possibilities. (This property is called "generic rigidity" in [3], as distinguished from other types of rigidity which we will not need here.)

For instance, let $G$ be the 4-gon, i.e., $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{12, 23, 34, 41\}$. $G$ is not rigid, since there are infinitely many incongruent rhombuses with equal side lengths.

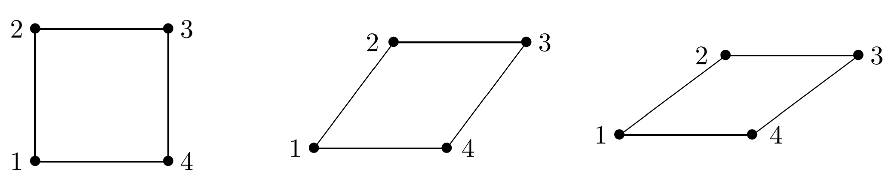


Figure 1. The 4-gon is not rigid

However, the graph $G' = (V, E \cup \{24\})$ is rigid, because a generic affine picture of $G'$ is determined by the lengths of its edges, up to isometries of $\mathbb{A}^2$ and finitely many possibilities.

**Definition 4.1.** The *length-rigidity matroid* $\mathcal{M}(V)$ (called the *2-dimensional generic rigidity matroid* in [3]) is the algebraic dependence matroid on the squares of lengths of edges

$$(x_v - x_w)^2 + (y_v - y_w)^2, \qquad v, w \in V. \tag{19}$$

We may regard $\mathcal{M}(V)$ as a matroid on $K(V)$, associating the polynomial of (19) with the edge $vw$. Accordingly, we say that a set of edges is independent in $\mathcal{M}(V)$, or *rigidity-independent*, if and only if the corresponding set of squared lengths is algebraically independent over $\mathbb{Q}$. Thus an edge set $E$ is rigid if and only if $E$ is a spanning set of $\mathcal{M}(V)$.

A fundamental result of rigidity theory is the following characterization of the independent sets and bases of $\mathcal{M}(V)$ [3, Theorem 4.2.1], originally due to G. Laman. An edge set $E \subset K(V)$ is rigidity-independent if and only if it satisfies *Laman's condition*:

$$|F| \leq 2|V(F)| - 3 \text{ for all } F \subset E. \tag{20}$$

Furthermore, a rigidity-independent set $E$ is a basis of $\mathcal{M}(V)$ if and only if

$$|E| = 2|V| - 3. \tag{21}$$

In addition, $E$ is a *rigidity circuit*—a minimal dependent set of $\mathcal{M}(V)$—if and only if $|E| = 2|V(E)| - 2$ and every proper subset $F$ of $E$ satisfies Laman's condition (20) [3, Theorem 4.3.1].

The rigidity circuits (called "rigidity cycles" in [3]) may be described another way. Define a *rigidity pseudocircuit* to be an edge set $E$ equal to the edge-disjoint union of two spanning trees of $V(E)$. Then a rigidity circuit is a minimal rigidity pseudocircuit [3, Lemma 4.9.3 and Theorem 4.9.1].

**Example 4.2.** Let $r \geq 3$. The *r-wheel* is the graph $W_r$ with vertices

$$\{v_0, v_1, \ldots, v_r\}$$

and edges

$$\{v_1v_2, v_2v_3, \ldots, v_rv_1\} \cup \{v_0v_1, v_0v_2, \ldots, v_0v_r\}.$$

For all $r \geq 3$, $W_r$ is a rigidity circuit [3, Exercise 4.13]. (In fact, $W_3 \cong K_4$ and $W_4$ are the only rigidity circuits on 5 or fewer vertices.) On the other hand, let $G'$ be the following graph:

$E(G')$ is a rigidity pseudocircuit, since it is the disjoint union of the spanning trees $\{12, 23, 34, 45\}$ and $\{13, 14, 24, 35\}$, but it is not a rigidity circuit since it contains $K_4$ as a proper subgraph.

**Definition 4.3.** Let $G = (V, E)$ be a rigidity pseudocircuit. A *special spanning tree* of $G$ is an edge set $T \subset E$ such that both $T$ and $T' = E \setminus T$ are spanning trees of $V$. The set of special spanning trees of $G$ is denoted by $\mathrm{CT}(G)$. The pair $T, T'$ is called a *2-tree decomposition* of $E$ (or of $G$).

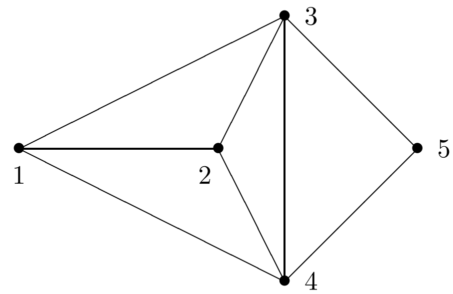


FIGURE 2. A pseudocircuit which is not a circuit

The special spanning trees of a rigidity circuit will play a fundamental role in describing the equations which define $\tilde{\mathcal{V}}(G)$ and $\tilde{\mathcal{S}}(G)$.

Our local affine coordinates on $\tilde{\mathcal{X}}(G)$ measure the slopes of edges rather than their lengths, leading to an alternate notion of rigidity.

**Definition 4.4.** The *slope-rigidity matroid* $\mathcal{M}^s(V)$ on $K(V)$ is the algebraic dependence matroid on the rational functions

$$m_{vw} = \frac{y_w - y_v}{x_w - x_v}.$$

**Theorem 4.5.** *Let $G = (V, E)$. The following are equivalent:*

(i) *$E$ is independent in $\mathcal{M}(V)$;*

(ii) *$E$ is independent in $\mathcal{M}^s(V)$;*

(iii) *$\tilde{\mathcal{S}}(G) = \operatorname{Spec} R_G$ $(\cong \mathbb{A}^{|E|})$;*

(iv) *$\mathcal{V}(G) = \mathcal{X}(G)$.*

*Proof.* Let $n = |V|$ and $r = |E|$.

$(i) \implies (iv)$: Since $\mathcal{X}(G)$ is defined locally by $2r$ equations among $2n + 2r$ coordinates, we have

$$\dim X \geq 2n = \dim \mathcal{V}(G)$$

for every irreducible component $X$ of $\mathcal{X}(G)$. Therefore $\mathcal{V}^\circ(G)$ is dense in $\mathcal{X}(G)$ if and only if every nondiscrete cellule has dimension $< 2n$.

Suppose $E$ is rigidity-independent, hence satisfies Laman's condition (20). Let $\mathcal{A}$ be a partition of $V$ which is not the discrete partition. The blocks of $\mathcal{A}$ may be numbered $A_1, \ldots, A_s$ so that

$$|A_1| = \cdots = |A_t| = 1 \qquad \text{and} \qquad |A_i| > 1 \text{ for } t < i \leq s.$$

We may rewrite the cellule dimension formula (14) as

$$\dim \mathcal{X}_{\mathcal{A}}(G) \;=\; 2s \;+\; \sum_{i=1}^{s} |K(A_i) \cap E|. \tag{22}$$

If $i \leq t$, then $K(A_i) = \emptyset$, while if $i > t$, then $|K(A_i) \cap E| \leq 2|A_i| - 3$ by Laman's condition (20). Hence

$$\begin{aligned} \dim \mathcal{X}_{\mathcal{A}}(G) \leq \ & 2s + \sum_{i=t+1}^{s} (2|A_i| - 3) \ = \ 2s + (2(n-t) - 3(s-t)) \\ = \ & 2n - s + t \ < \ 2n. \end{aligned} \tag{23}$$

as desired.

$(iv) \implies (iii)$: No nonzero element of $R_G$ vanishes on $\tilde{\mathcal{X}}(G)$, since the projection of the indiscrete cellule $\tilde{\mathcal{X}}_{\mathcal{I}}(G)$ on the second factor in (1) is surjective. On the other hand, every element of $R_G$ that vanishes on $\tilde{\mathcal{S}}(G)$ vanishes on $\tilde{\mathcal{V}}(G)$. We conclude that if (iii) fails, then (iv) fails as well.

$(iii) \implies (ii)$: This is essentially the definition of the slope-rigidity matroid.

$(ii) \implies (i)$: Suppose that $E$ is independent in $\mathcal{M}^s(V)$. Let $F \subset E$, and let $H$ be the graph $(V(F), F)$. Then $\dim \tilde{\mathcal{V}}(H) = 2|V(F)|$, and all fibers of the canonical surjection $\tilde{\mathcal{V}}(H) \to \tilde{\mathcal{S}}(H)$ have dimension $\geq 3$ (since translation and scaling do not affect slope), whence

$$\dim \tilde{\mathcal{S}}(H) \leq 2|V(F)| - 3. \tag{24}$$

On the other hand, $F$ is independent in $\mathcal{M}^s(V)$, so $\{m_f \ | \ f \in F\}$ is algebraically independent, and these variables form a system of parameters for $\tilde{\mathcal{S}}(H)$, so

$$|F| \leq \dim \tilde{\mathcal{S}}(H). \tag{25}$$

Together, (24) and (24) imply Laman's condition (20) for $E$. □

We have recovered the following theorem: [1]

**Corollary 4.6.** *For every vertex set $V$, the length-rigidity matroid $\mathcal{M}(V)$ and the slope-rigidity matroid $\mathcal{M}^s(V)$ are equal.*

## 5. Equations Defining $\tilde{\mathcal{V}}(G)$

Let $G = (V, E)$ be a connected graph. In this section, we explicitly construct an ideal $I = I_G$ defining the affine picture and affine slope varieties of $G$ set-theoretically. The generators of $I$ turn out to have an elegant combinatorial description: their terms enumerate special spanning trees of the rigidity circuit subgraphs of $G$.

We begin with some computations which are most conveniently expressed in terms of the homology of $G$, considered as a 1-dimensional simplicial complex.

A *directed edge* of $G$ is a symbol $[v, w]$, where $vw \in E$. An *orientation* of an edge $e = vw$ is chosen by putting either $e = [v, w]$ or $e = [w, v]$. In addition, for

[1] During the refereeing process, the author learned that this result had already appeared in the literature: see, e.g., [9, Corollary 4.1.3]. The author wishes to thank Walter Whiteley for helpful conversations on this topic.

$e = [v, w] \in E$, we define

$$x_f = x_w - x_v. \tag{26}$$

In what follows, we fix an arbitrary orientation for each edge of $G$.

Let $C$ be the free $\mathbb{Z}$-module on the directed edges of $G$, modulo the relations

$$[w, v] = -[v, w].$$

Homologically, $C$ is the group of 1-chains. The subgroup of 1-cycles is

$$Z = \left\{ \sum_e c_e e \in C \;\mid\; \sum_e c_e x_e = 0 \right\}.$$

Note that $Z$ is generated by the cycles

$$z(P) = \sum_{i=1}^{s} [v_i, v_{i+1}] \tag{27}$$

where $P = (v_1, \dots, v_s, v_{s+1} = v_1)$ is a polygon of $G$.

The *support* of a chain is defined by

$$\operatorname{supp}\left( \sum_{e \in E} c_e e \right) = \{ e \in E \;\mid\; c_e \neq 0 \} \;. \tag{28}$$

Note that if $\gamma \in Z$ and $\operatorname{supp}(\gamma) \subset T$ for some tree $T$, then $\gamma = 0$.

Let $T$ be a spanning tree of $G$ and $S = E \setminus T$. Fix an orientation for each edge of $E$. For each edge $e = [v, w] \in S$, the edge set $T \cup \{e\}$ contains a unique polygon of the form

$$P_T(e) = (v = v_1, \dots, v_s = w, v). \tag{29}$$

There is a corresponding cycle

$$\begin{aligned} z_T(e) &= [v_1, v_2] + \cdots + [v_{s-1}, v_s] + [v_s, v_1] && (30)\\ &= -[v, w] + \sum_{i=1}^{s-1} [v_i, v_{i+1}] && (31)\\ &= -e + \sum_{f \in T} c_{ef}^T \; f && (32) \end{aligned}$$

where $c_{ef}^T \in \{0, 1, -1\}$ for all $f$. Note that for every spanning tree $T$ of $G$, the set

$$\{ z_T(e) \;\mid\; e \in T \}$$

generates $Z$. Indeed, if $\zeta = \sum_{e \in E} b_e e$ is a cycle, then

$$\zeta' = \zeta + \sum_{e \in E \setminus T} b_e z_T(e)$$

is a cycle with support contained in $T$, so $\zeta' = 0$.

There is an injective map of $\mathbb{Z}$-modules $C \to R'_G$ sending

$$[v, w] \mapsto m_{vw}(x_v - x_w) \tag{33}$$

for all directed edges $[v,w]$. The image of $Z$ under this map contains all polynomials $L(P)$ defined in (8). Therefore, for every spanning tree $T$, the set $\{L(P_T(e)) \mid e \in S\}$ generates an ideal defining $\tilde{\mathcal{X}}(G)$ set-theoretically.

Let $e=[a,b]\in S$, and let $P_T(e)$ be the polygon of (29). Then

$$\begin{aligned} L(P_T(e)) &= \left(\sum_{i=1}^{s-1} m_{e_i}x_{e_i}\right) + m_e x_e \\ &= -\sum_{f\in T} c^T_{ef} m_f x_f + m_e \sum_{f\in F} c^T_{ef} x_f \\ &= \sum_{f\in T} c^T_{ef}(m_e - m_f)x_f \end{aligned} \tag{34}$$

Collecting the equations (34) for all edges of $S$, we obtain a matrix equation

$$\Big[L(P_T(e))\Big]_{e\in S} \;=\; M_T X_T \;=\; \left(D_S C_T - C_T D_T\right) X_T \tag{35}$$

where

$$\begin{aligned} C_T &= \left[c^T_{ef}\right]_{e\in S, f\in T}; \\ M_T &= \left[c^T_{ef}(m_e - m_f)\right]_{e\in S, f\in T}; \\ D_T &= \text{diagonal matrix with entries } m_f,\ f\in T; \\ D_S &= \text{diagonal matrix with entries } m_e,\ e\in S; \\ X_T &= \text{column vector } \left[x_f\right]_{f\in T}. \end{aligned} \tag{36}$$

Moreover, the equations (7) defining $\mathcal{X}(G)$ are equivalent to the single matrix equation

$$M_T X_T = 0. \tag{37}$$

**Example 5.1.** Let $G=K_4$. Orient each edge $ab\in E(G)$ as $[a,b]$, where $a<b$. Let $T=\{[1,2],[1,3],[1,4]\}$. For $e=[v,w]\notin T$, we have

$$P_T(e) = [v,w]+[w,1]+[1,v] = [v,w] - \left([1,w]-[1,v]\right),$$

so the matrix $C_T$ is

$$\begin{array}{cc|c} & & f \\ & & \begin{array}{ccc} [1,2] & [1,3] & [1,4] \end{array} \\ e & \begin{array}{c} [2,3] \\ [2,4] \\ [3,4] \end{array} & \left[\begin{array}{ccc} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}\right], \end{array} \tag{38}$$

the polynomials of (34) are

$$L\left(P_T\left([a,b]\right)\right) = m_{ab}(x_b - x_a) + m_{1b}(x_b - x_a) + m_{1b}(x_a - x_b) \tag{39}$$

for $[a,b] \in S = \{[2,3],[2,4],[3,4]\}$, and the matrix of (35) is

$$M_T X_T \;=\; \begin{bmatrix} m_{12}-m_{23} & m_{23}-m_{13} & 0 \\ m_{12}-m_{24} & 0 & m_{24}-m_{14} \\ 0 & m_{13}-m_{34} & m_{34}-m_{14} \end{bmatrix} \begin{bmatrix} x_2-x_1 \\ x_3-x_1 \\ x_4-x_1 \end{bmatrix}. \tag{40}$$

**Lemma 5.2.** *Let $G=(V,E)$ be a rigidity circuit. Let $T \in \mathrm{CT}(G)$ a special spanning tree, and $S = E \setminus T$. (Recall that $S$ is also a spanning tree of $G$, and that $M_T$, $C_T$, and $C_S$ are $(|V|-1)\times(|V|-1)$ square matrices.) Then $C_T = C_S^{-1}$.*

*Proof.* Replacing each edge $f$ on the right side of (32) with $f + z_S(f)$, we see that

$$e - \sum_{f\in T} c^T_{ef} \sum_{g\in S} c^S_{fg} g \;=\; e - \sum_{g\in S} g \sum_{f\in T} c^T_{ef} c^S_{fg} \;\in\; Z. \tag{41}$$

The support of this cycle lies in $S$, so it is zero. Hence, for all $e,g \in S$,

$$\sum_{f\in T} c^T_{ef}\; c^S_{fg} \;=\; \delta_{eg}, \tag{42}$$

(where $\delta_{eg}$ is the Kronecker delta), which is the statement that $C_T = C_S^{-1}$. □

Somewhat more generally, if $G$ is a rigidity pseudocircuit and $T,U$ are two spanning trees of $G$, then every member of the set $\{L(P_U(e)) \mid e \in E\setminus U\}$ may be expressed as an integer linear combination of the polynomials $\{L(P_T(e)) \mid e \in E\setminus T\}$, and vice versa. In addition $\{L(P_T(e)) \mid e \in E\setminus T\}$ is linearly independent, since each variable $m_e$, for $e \notin T$, appears in exactly one $L(P_T(e))$. Similarly, $\{L(P_U(e)) \mid e \in E\setminus U\}$ is linearly independent. Therefore

$$M_T X_T = B M_U X_U$$

for some invertible integer matrix $B$. In particular $\det B = \pm 1$, so the polynomial $\det M_T$ is independent, up to sign, of the choice of $T$. This motivates the following definition:

**Definition 5.3.** Let $G$ be a rigidity pseudocircuit. The *tree polynomial* of $G$ is defined up to sigh as

$$\tau(G) = \det M_T \tag{43}$$

where $M_T$ is the matrix of (35) and (36).

The name "tree polynomial" is justified by the following theorem. One more piece of notation: to each edge set $F \subset E$, we associate the squarefree monomial

$$m_F = \prod_{f\in F} m_f. \tag{44}$$

**Theorem 5.4.** *Let $G=(V,E)$ be a rigidity circuit, $|V|=n$, and $|E|=2n-2$.*

(i) *The polynomial $\tau(G)$ is homogeneous of degree $n-1$ and squarefree (i.e., a sum of squarefree monomials). That is, we may write*

$$\tau(G) = \sum_{\substack{F\subset E \\ |F|=n-1}} \varepsilon(F) m_F. \tag{45}$$

(ii) *For all $F \subset E$ with $|F| = n-1$, we have*

$$\varepsilon(F) = (-1)^{n-1}\varepsilon(E \setminus F). \tag{46}$$

(iii) *For all $F \subset E$ with $|F| = n-1$, the coefficient $\varepsilon(F)$ is $\pm 1$ if $F$ is a special spanning tree, otherwise 0. That is,*

$$\tau(G) \;=\; \sum_{T \in \mathrm{CT}(G)} \varepsilon(T) m_T \tag{47}$$

*where $\varepsilon(T) \in \{1, -1\}$.*

(iv) *$\tau(G)$ vanishes on $\tilde{\mathcal{V}}(G)$.*

(v) *$\tau(G)$ vanishes on $\tilde{\mathcal{S}}(G)$.*

(vi) *$\tau(G)$ is irreducible.*

*Proof.* (i) By construction, $\tau(G)$ is homogeneous of degree $n-1$. Let $e \in E$. If $e \in T$ then $m_e$ appears in only one column of $M_T$, while if $e \in S$ then $m_e$ appears in only one row of $M_T$. It follows that $\tau(G)$ is squarefree.

(ii) Since no $m_e$ appears in more than one row or in more than one column, each nonzero term in the determinant expansion of $\tau(G)$ is of the form

$$(m_{1,1} - m_{1,2}) \dots (m_{n-1,1} - m_{n-1,2})$$

where the $m_j$ are all distinct. This may be expressed as a sum of binomials of the form

$$m_T + (-1)^{n-1} m_{E \setminus T},$$

from which (46) follows.

(iii) Let $T$ be a spanning tree of $G$ and $S = E \setminus T$. By the construction of $\tau(G)$ (in particular (35)), we have

$$\varepsilon(T) = (-1)^{n-1} \det C_T \qquad \text{and} \qquad \varepsilon(S) = \det C_T.$$

If $T, S \in \mathrm{CT}(G)$, then $C_S = C_T^{-1}$ by Lemma 5.2; in particular $\det C_T = \det C_S = \pm 1$ (since $C_S$ and $C_T$ have integer entries).

Now suppose that $F \subset E$ has cardinality $n-1$, but $F$ is not a special spanning tree. Then $F$ and $E \setminus F$ are not both trees. Because of the result of (ii), we may assume without loss of generality that $F$ is not a tree.

Let $A \subset F$ be a minimal set of edges such that $F \setminus A$ is a forest (note that $A \neq \emptyset$). Let $T$ be a spanning tree of $G$ containing $F \setminus A$; then

$$T \cap (E \setminus F) = T \setminus (F \setminus A) \neq \emptyset. \tag{48}$$

Let $S = E \setminus T$ ($\supset A$). Construct the matrix $M_T$ as in (35). Note that

$$a \in A, b \in T \cap (E \setminus F) \implies c^T_{ab} = 0 \tag{49}$$

since the unique circuit of $T \cup \{a\}$ is contained in $F \setminus A \cup \{a\}$.

Let $a \in A$. By (49), every entry of the row of $M_T$ corresponding to $a$ is either zero or of the form $\pm(m_a - m_f)$, where $f \in F \setminus A$. In particular, no variable dividing $m_{E \setminus F}$ appears in that row. Hence $\varepsilon(E \setminus F) = 0$, and $\varepsilon(F) = 0$ by (46).

(iv) Since the generic affine pictures are dense in $\tilde{\mathcal{V}}(G)$, it suffices to show that $\tau(G)$ vanishes at each $\mathbf{P} \in \mathcal{V}^\circ(G) \cap \tilde{\mathcal{V}}(G)$. Indeed, $M_T(\mathbf{P})X_T(\mathbf{P}) = 0$ and $X_T(\mathbf{P}) \neq 0$, so $\tau(G) = \det M_T$ vanishes at $\mathbf{P}$.

(v) This is immediate from (iv) and the definition of $\tilde{\mathcal{S}}(G)$.

(vi) Suppose that $\tau(G) = f_1 \cdot f_2$. For every $e \in E$, we have

$$\deg_{m_e}(\tau(G)) = \deg_{m_e}(f_1) + \deg_{m_e}(f_2) = 1,$$

so $E$ is a disjoint union $E_1 \cup E_2$, where $E_i = \{e \in E \mid \deg_{m_e}(f_i) = 1\}$. Let $G_i = (V, E_i)$. Since $\tilde{\mathcal{S}}(G)$ is by definition irreducible, either $f_1$ or $f_2$ vanishes on $\tilde{\mathcal{S}}(G)$ by part (v). Assume without loss of generality that $f_1$ vanishes on $\tilde{\mathcal{S}}(G)$. Then $f_1$ vanishes on $\tilde{\mathcal{S}}(G_1)$ as well via the natural surjection $\tilde{\mathcal{S}}(G) \to \tilde{\mathcal{S}}(G_i)$. By Theorem 4.5, $E_1$ must be rigidity-dependent, so $E_1 = E$ (because $E$ contains no proper rigidity-dependent subset). Therefore $E_2 = \emptyset$ and the factorization of $\tau(G)$ is trivial. $\square$

**Remark 5.5.** Given a connected graph $G = (V, E)$ with $|E| = 2|V| - 2$, not necessarily a rigidity circuit, and a spanning tree $T \subset E$, we may construct the matrix $M_T$ and define $\tau(G) = \det M_T$ as before. If $G$ is not a rigidity pseudocircuit, then the proof of Theorem 5.4 implies that $\tau(G) = 0$. If $G$ is a rigidity pseudocircuit but not a circuit—say $G$ contains a rigidity circuit $G' = (V', E')$ as a proper subgraph—then $\tau(G)$ is well-defined, up to sign, over all choices of $T$, and Theorem 5.4 goes through as before, with the exception that $\tau(G)$ is not irreducible. Indeed, let $T'$ be a spanning tree of $G'$ and $T \supset T'$ a spanning tree of $G$. Put $S = E \setminus T$ and $S' = E' \setminus T'$. Then the matrix $M_T$ has the form

$$\begin{bmatrix} M_{T'} & 0 \\ * & * \end{bmatrix}$$

where the $|V'| - 1$ uppermost rows correspond to edges in $S'$ and the $|V'| - 1$ leftmost columns correspond to edges in $T'$. It follows that $\tau(G')$ is a proper divisor of $\tau(G)$.

**Example 5.6.** Let $G = K_4$. Let $T$, $M_T$, $X_T$ be as in Example 5.1. There are two kinds of spanning trees of $G$: paths $(a, b, c, d)$, and "stars," such as $T$. The paths are special; the stars are not. There are $4!/2 = 12$ paths, and the sign of a path is given by the sign of the corresponding permutation in the symmetric group $S_4$, that is,

$$\tau(G) = \det M_T = -\frac{1}{2} \sum_{\sigma \in S_4} sgn(\sigma) m_{\sigma_1 \sigma_2} m_{\sigma_2 \sigma_3} m_{\sigma_3 \sigma_4} \ . \tag{50}$$

On the other hand, if $G'$ is the graph of Example 4.2 (a rigidity pseudocircuit which is not a circuit), then

$$\tau(G') = \pm\,(m_{35} - m_{45})\,\tau(G).$$

**Theorem 5.7.** *Let $G = (V, E)$ be a graph. Let $I = I_G$ be the ideal of $R_G$ generated by all tree polynomials $\tau(C)$, where $C$ is a rigidity circuit subgraph of $G$. Then:*

(i) *$\tilde{\mathcal{V}}(G)$ is the vanishing locus of $IR'_G$ in $\tilde{\mathcal{X}}(G)$.*

(ii) *$\tilde{\mathcal{S}}(G)$ is the vanishing locus of $I$ in* $\operatorname{Spec} R_G$.

*Proof.* We may assume without loss of generality that $G$ is connected, since every rigidity circuit is connected and $\tilde{\mathcal{V}}(G)$ is the product of the picture varieties of its connected components. Let $n = |V|$, $r = |E|$.

(i) Let $Y$ be the vanishing locus of $IR'_G$ in $\tilde{\mathcal{X}}(G)$. For each rigidity circuit subgraph $C$ of $G$, the tree polynomial $\tau(C)$ vanishes on $\tilde{\mathcal{V}}(C)$ by Theorem 5.4, so it vanishes on $\tilde{\mathcal{V}}(G)$ as well. Hence $\tilde{\mathcal{V}}(G) \subset Y$.

We now establish the reverse inclusion, proceeding by induction on $n$. By Theorem 4.5, there is nothing to prove when $E$ is rigidity-independent, in particular when $n \leq 3$.

Let $\mathbf{P} \in Y \cap \mathcal{X}_{\mathcal{A}}(G)$, where $\mathcal{A} = \{A_1, \dots, A_s\}$ is a partition of $V$ with $s$ parts. We wish to show that $\mathbf{P} \in \tilde{\mathcal{V}}(G)$.

*Case 1: $s = n$.* Here $\mathcal{A}$ is the discrete partition, so $\mathbf{P} \in \tilde{\mathcal{V}}(G)$ by definition.

*Case 2: $2 \leq s \leq n-1$.* For $1 \leq i \leq s$, let

$$G_i = (A_i, E \cap K(A_i)) \tag{51}$$

(a subgraph of $G$), and let

$$U = \bigcup_{\mathcal{B} \preceq \mathcal{A}} \tilde{\mathcal{X}}_{\mathcal{B}}(G) = \left\{ \mathbf{P}' \in \tilde{\mathcal{X}}(G) \;\mid\; \mathbf{P}'(v) \neq \mathbf{P}'(w) \text{ if } v \not\sim_{\mathcal{A}} w \right\}, \tag{52}$$

an open subset of $\tilde{\mathcal{X}}(G)$ containing $\mathbf{P}$. There is a natural open embedding

$$\pi : U \to \prod_{i=1}^{s} \tilde{\mathcal{X}}(G_i). \tag{53}$$

Note that $I_{G_i} \subset IR_G$ for all $i$. By induction, $\tilde{\mathcal{V}}(G_i)$ is the vanishing locus of $I_{G_i}$ in $\tilde{\mathcal{X}}(G_i)$. Therefore

$$\mathbf{P} \in \pi^{-1} \left( \prod_{i=1}^{s} \tilde{\mathcal{V}}(G_i) \right) . \tag{54}$$

This set is irreducible and contains $\mathcal{V}^\circ(G)$ as an open, hence dense, subset. Therefore $\mathbf{P} \in \tilde{\mathcal{V}}(G)$ as desired.

*Case 3: $s = 1$.* That is, $\mathcal{A}$ is the indiscrete partition of $V$. Fix a spanning tree $T$ of $G$ and let $M_T$ be the matrix defined in (35). Recall that $M_T$ is an $(r-n+1) \times (n-1)$. The rows and columns of $M_T$ are indexed by the edges of $E \setminus T$ and $T$, respectively. In addition, $\tilde{\mathcal{X}}(G)$ is defined by the matrix equation (37), and $X_T(\mathbf{P})$ is the zero matrix.

We claim that

$$\operatorname{rank}(\mathbf{P}) < n-1. \tag{55}$$

If $M_T$ has fewer than $n-1$ rows then there is nothing to prove. Otherwise, let $M'$ be any $(n-1)\times(n-1)$ submatrix $M'$ of $M_T$, with rows indexed by the elements of some edge set $S \subset E \setminus T$. Then $|T \cup S| = 2n-2$, so $|T \cup S|$ does not satisfy Laman's condition (20) and must contain some rigidity circuit $C$. By Remark 5.5, $\tau(C)$ divides $\det M'$, establishing (55).

It follows from (55) that the nullspace of $M_T(\mathbf{P})$ contains a nonzero vector $X'$. For every $\lambda \in \mathbf{k}$, we have $(M_T)(\lambda X') = 0$, so there is a picture $\mathbf{P}_\lambda$ with the same slope coordinates as $\mathbf{P}$ and $x$-coordinates of vertices given by $\lambda X'$. The $\mathbf{P}_\lambda$ form an affine line in $Y$ with $\mathbf{P}_0 = \mathbf{P}$. Moreover, if $\lambda \neq 0$, then $\mathbf{P}_\lambda \notin \mathcal{X}_{\mathcal{A}}(G)$, hence $\mathbf{P}_\lambda \in \tilde{\mathcal{V}}(G)$ by the previous two cases. Therefore $\mathbf{P}_0 = \mathbf{P} \in \tilde{\mathcal{V}}(G)$ as well.

(ii) Let $Z$ be the vanishing locus of $I$ in $\operatorname{Spec} R_G$. It is immediate from Definition 3.8 that $Z \supset \tilde{\mathcal{S}}(G)$. Now suppose that $\mathbf{m} \in Z$, i.e., $\mathbf{m}$ is an affine slope picture at which all tree polynomials vanish. Fix a spanning tree $T$ of $G$ and let $X$ be a nullvector of the matrix $M_T(\mathbf{m})$. Together, $\mathbf{m}$ and $X$ define an affine line in $\tilde{\mathcal{X}}(G)$; by part (i) of the theorem, the line is contained in $\tilde{\mathcal{V}}(G)$. Therefore $\mathbf{m} \in \tilde{\mathcal{S}}(G)$. $\square$

We have proven that

$$\tilde{\mathcal{V}}(G) \cong \mathbb{A}^1 \times \operatorname{Spec} R'_G/\sqrt{J_G} \tag{56}$$

and

$$\tilde{\mathcal{S}}(G) \cong \operatorname{Spec} R_G/\sqrt{I_G} \tag{57}$$

as reduced schemes, where $J_G = I_G R'_G + (L(P))$. However, we do not yet know whether the ideals $J_G$ and $I_G$ are radical. In the special case that $G$ is a rigidity cycle, the ideal $I_G$ is radical because it is principal, generated by the irreducible polynomial $\tau(G)$. We will prove in a separate paper that $I_G$ is radical when $G$ is the complete graph $K_n$.

## 6. Geometric Properties of $\mathcal{X}(G)$ and $\mathcal{V}(G)$

In this section, we use the algebraic results of the previous sections to prove certain geometric facts about the picture space. First, we give a combinatorial condition which describes when one cellule of $\mathcal{X}(G)$ is contained in the closure of another cellule. Using this result, we can give a complete combinatorial description of the irreducible components of the picture space. Second, we present an inductive criterion on $G$ which implies that $\mathcal{V}(G)$ is Cohen-Macaulay; one consequence of this result is that $\mathcal{V}(G)$ is Cohen-Macaulay whenever $G$ is rigidity-independent.

**Definition 6.1.** Let $G = (V, E)$ be a graph, $F \subset E$, and $\mathcal{A}$ a partition of $V$. We say $\mathcal{A}$ *collapses* $F$ if all vertices of $V(F)$ are contained in the same block of $\mathcal{A}$. In this case, the equations defining $\mathcal{X}(G)$ impose no restrictions on the slopes of the lines $\mathbf{P}(e)$ for pictures $\mathbf{P} \in \mathcal{X}_{\mathcal{A}}(G)$ and edges $e \in F$.

**Theorem 6.2.** *Let $G = (V, E)$ be a graph and $\mathcal{A}$ a partition of $V$. Then $\mathcal{X}_{\mathcal{A}}(G) \subset \mathcal{V}(G)$ if and only if no rigidity circuit of $G$ is collapsed by $\mathcal{A}$.*

*Proof.* We first consider two special cases. If $E$ is rigidity-independent, then by Theorem 4.5 we have $\mathcal{V}(G) = \mathcal{X}(G)$, so there is nothing to prove. Now suppose that $E$ is a rigidity circuit. The only partition of $V$ which collapses $E$ is the indiscrete partition $\mathcal{I}$. The cellule dimension formula (14) gives $\dim \mathcal{X}_{\mathcal{I}}(G) = 2n = \dim \mathcal{V}^\circ(G)$, so $\mathcal{X}_{\mathcal{I}}(G) \not\subset \overline{\mathcal{V}^\circ(G)} = \mathcal{V}(G)$. On the other hand, if $\mathcal{A}$ is neither the discrete nor indiscrete partition, then $\dim \mathcal{X}_{\mathcal{A}}(G) < 2n$ by (23) (since Laman's condition (20) holds for every proper subset of a rigidity circuit). The indiscrete cellule is itself closed, and all components of $\mathcal{X}(G)$ have dimension $\geq 2n$, so $\mathcal{X}_{\mathcal{A}}(G) \subset \mathcal{V}(G)$. (Indeed, we can now see that the picture space of a rigidity circuit $G$ has two irreducible components, namely $\mathcal{V}(G)$ and the indiscrete cellule.)

We turn now to the general case. By Theorem 5.7, it is enough to prove that for every rigidity circuit $C$ of $G$, $\tau(C)$ vanishes on $\tilde{\mathcal{X}}_{\mathcal{A}}(G)$ if and only if $\mathcal{A}$ does not collapse $C$. One direction is immediate: if $\mathcal{A}$ collapses $C$, then $\tau(C)$ does not vanish on $\tilde{\mathcal{X}}_{\mathcal{A}}(G)$ and consequently $\tilde{\mathcal{X}}_{\mathcal{A}}(G) \not\subset \tilde{\mathcal{V}}(G)$. On the other hand, suppose that $\mathcal{A}$ does not collapse $C$. Consider the natural map $\tilde{\mathcal{X}}_{\mathcal{A}}(G) \to \tilde{\mathcal{X}}(C)$. The image of this map does not intersect the indiscrete cellule of $\tilde{\mathcal{X}}(C)$. By the special case, $\tau(C)$ vanishes on the image, hence on $\mathcal{X}_{\mathcal{A}}(G)$. $\square$

Given a graph $G = (V, E)$ and a partition $\mathcal{A} = \{A_1, \ldots, A_s\}$ of $V$, let $G/\mathcal{A}$ denote the graph whose vertices are the blocks of $\mathcal{A}$ and whose edges are

$$\big\{\{A_i, A_j\} \ \mid \ vw \in E \text{ for some } v \in A_i,\ w \in A_j\big\}.$$

Also, if $\mathcal{A}$ and $\mathcal{B}$ are partitions of $V$ with $\mathcal{A} \preceq \mathcal{B}$, then we write $\mathcal{B}/\mathcal{A}$ for the partition on the blocks of $\mathcal{A}$ setting two blocks equivalent if both are subsets of the same block of $\mathcal{B}$.

**Theorem 6.3.** *Let $G = (V, E)$ be a graph, and let $\mathcal{A}, \mathcal{B}$ be partitions of $V$. Then $\mathcal{X}_{\mathcal{B}}(G) \subset \overline{\mathcal{X}_{\mathcal{A}}(G)}$ if and only if the following conditions hold:*

*(a) $\mathcal{A} \preceq \mathcal{B}$;*

*(b) No rigidity circuit of $G/\mathcal{A}$ is collapsed by $\mathcal{B}/\mathcal{A}$; and*

*(c) If $A_i$ and $A_j$ are distinct blocks of $\mathcal{A}$ contained in the same block of $\mathcal{B}$, then $E$ contains at most one edge between $A_i$ and $A_j$ (i.e., with one endpoint in each set).*

*Proof.* It is sufficient to prove the corresponding statement for the affine cellules $\tilde{\mathcal{X}}_{\mathcal{B}}(G) = \mathcal{X}_{\mathcal{B}}(G) \cap \tilde{\mathcal{X}}(G)$ and $\tilde{\mathcal{X}}_{\mathcal{A}}(G) = \mathcal{X}_{\mathcal{A}}(G) \cap \tilde{\mathcal{X}}(G)$.

Suppose that $\mathcal{X}_{\mathcal{B}}(G) \subset \overline{\mathcal{X}_{\mathcal{A}}(G)}$. If $v \sim_{\mathcal{A}} w$, the equation $\mathbf{P}(v) = \mathbf{P}(w)$ holds on $\mathcal{X}_{\mathcal{B}}(G)$, so $v \sim_{\mathcal{B}} w$, establishing (a). For each rigidity circuit $C$ of $G/\mathcal{A}$, the function $\tau(C)$ vanishes on $\mathcal{X}_{\mathcal{B}}(G)$, so $\mathcal{B}/\mathcal{A}$ cannot collapse $C$. Finally, if $A_i$ and $A_j$ are contained in the same block of $\mathcal{B}$ and $E$ contains two distinct edges $e, e'$ between $A_i$ and $A_j$, then the equation $\mathbf{P}(e) = \mathbf{P}(e')$ holds on $\mathcal{X}_{\mathcal{A}}(G)$ but not on $\mathcal{X}_{\mathcal{B}}(G)$, a contradiction.

Now suppose that conditions (a), (b) and (c) hold. Let

$$\begin{aligned} E' &= \{vw \in E \mid v \sim_{\mathcal{A}} w\}, && (58)\\ U &= \mathbb{A}^{|E'|}, && (59)\\ Z &= \bigcup_{\mathcal{B} \succeq \mathcal{A}} \tilde{\mathcal{X}}_{\mathcal{B}}(G) = \left\{\mathbf{P} \in \tilde{\mathcal{X}}(G) \mid \mathbf{P}(v) = \mathbf{P}(w) \text{ if } v \sim_{\mathcal{A}} w\right\}. && (60) \end{aligned}$$

Observe that the data for an affine picture $\mathbf{P} \in Z$ is the same as that describing a picture of $G/\mathcal{A}$ together with the slopes of the lines $\mathbf{P}(e)$ for $e \in E'$. Hence we have an isomorphism

$$\pi : Z \to \tilde{\mathcal{X}}(G/\mathcal{A}) \times U, \tag{61}$$

Restricting $\pi$ to the cellules under consideration, we have a commutative diagram of quasiaffine varieties

$$\begin{array}{ccccccc} \tilde{\mathcal{X}}_{\mathcal{A}}(G) & \subset & \overline{\tilde{\mathcal{X}}_{\mathcal{A}}(G)} & \subset & Z & \supset & \tilde{\mathcal{X}}_{\mathcal{B}}(G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{V}}^{\circ}(G/\mathcal{A}) \times U & \subset & \tilde{\mathcal{V}}(G/\mathcal{A}) \times U & \subset & \tilde{\mathcal{X}}(G/\mathcal{A}) \times U & \supset & \tilde{\mathcal{X}}_{\mathcal{B}/\mathcal{A}}(G/\mathcal{A}) \times U \end{array} \tag{62}$$

where the vertical arrows are isomorphisms. This implies that

$$\tilde{\mathcal{X}}_{\mathcal{B}}(G) \subset \overline{\tilde{\mathcal{X}}_{\mathcal{A}}(G)} \iff \tilde{\mathcal{X}}_{\mathcal{B}/\mathcal{A}}(G/\mathcal{A}) \subset \tilde{\mathcal{V}}(G/\mathcal{A}) \tag{63}$$

which is in turn equivalent to condition (b) by Theorem 6.2. □

**Remark 6.4.** The notion of a pseudocircuit may be extended to multigraphs: a multigraph $(V, E)$ is called a *pseudocircuit* if $|E| = 2|V| - 2$ and $|F| \leq 2|V(F)| - 2$ for all $\emptyset \neq F \subset E$ [3, p. 118]. For instance, a double edge is a pseudocircuit. In the previous theorem, we may consider $G/\mathcal{A}$ as a multigraph, in which the multiplicity of an edge $\{A_i, A_j\}$ is the number of edges in $E$ with one endpoint in each of $A_i$ and $A_j$. Then conditions (b) and (c) together are equivalent to the single condition that $\mathcal{B}/\mathcal{A}$ collapse no multigraph pseudocircuit of $G/\mathcal{A}$.

An immediate consequence of Theorem 6.3 is the following characterization of the irreducible components of $\mathcal{X}(G)$.

**Theorem 6.5.** *Let $G = (V, E)$. Then the irreducible components of $\mathcal{X}(G)$ are exactly the subvarieties $\overline{\mathcal{X}_{\mathcal{A}}(G)}$, where $\mathcal{A}$ is maximal with respect to the partial order described in Theorem 6.3.* □

We next consider the Cohen-Macaulay property. Our main tool is the fact that if $X$ is a Cohen-Macaulay scheme and $Z$ is a "strongly Cohen-Macaulay" subscheme of $X$, then the blowup of $X$ along $Z$ is Cohen-Macaulay [4, Theorem 4.2] (see also [8]). In particular, a local complete intersection subscheme of a Cohen-Macaulay scheme is strongly Cohen-Macaulay.

**Lemma 6.6.** *Let $G = (V, E)$, $e = vw \in E$, and $H = (V, E \setminus \{e\})$. Suppose that $\mathcal{V}(H)$ is Cohen-Macaulay and that $\mathcal{V}(H) \cap \mathcal{X}_{\mathcal{A}}(H)$ has codimension $\geq 2$ in $\mathcal{V}(H)$ for all partitions $\mathcal{A}$ of $V$ with $v \sim_{\mathcal{A}} w$. Then $\mathcal{V}(G)$ is Cohen-Macaulay.*

*Proof.* Let $Z$ be the (possibly non-reduced) intersection $\mathcal{V}(H) \cap C_e(H)$. $Z$ is defined in local affine coordinates by two equations, namely $x_v = x_w$ and $y_v = y_w$, so each of its components has codimension $\leq 2$. On the other hand, $C_e(H)$ is set-theoretically the union of cellules $\mathcal{X}_A(H)$ with $v \sim_{\mathcal{A}} w$. Therefore

$$\operatorname{codim} Z \geq \operatorname{codim} C_e(H) \geq 2. \tag{64}$$

In particular, $Z$ is a local complete intersection in $\mathcal{V}(H)$, and $\mathcal{V}(G)$ is the blowup of $\mathcal{V}(H)$ along $Z$, so $\mathcal{V}(G)$ is Cohen-Macaulay. □

**Theorem 6.7.** *Let $G = (V, E)$, $e = vw \in E$, and $H = (V, E \setminus \{e\})$. If $\mathcal{V}(H)$ is Cohen-Macaulay and $e$ is not contained in any rigidity circuit subgraph of $G$, then $\mathcal{V}(G)$ is Cohen-Macaulay.*

*Proof.* Let $\mathcal{A}$ be a partition of $V$ with $v \sim_{\mathcal{A}} w$. The cellule $\mathcal{V}_{\mathcal{A}}(G) = \mathcal{X}_{\mathcal{A}}(G) \cap \mathcal{V}(G)$ has codimension $\geq 1$ in $\mathcal{V}(G)$. Since no rigidity circuit contains $e$, the equations defining $\mathcal{V}_{\mathcal{A}}(G)$ impose no constraints on the line $\mathbf{P}(e)$. Therefore

$$\mathcal{V}_{\mathcal{A}}(G) \cong \mathcal{V}_{\mathcal{A}}(H) \times \mathbb{P}^1.$$

In particular $\mathcal{V}_{\mathcal{A}}(H)$ has codimension $\geq 2$ in $\mathcal{V}(H)$, since $\dim \mathcal{V}(G) = \dim \mathcal{V}(H) = 2|V|$. Thus $\mathcal{V}(G)$ is Cohen-Macaulay by Lemma 6.6.

**Theorem 6.8.** *Let $G = (V, E)$. If $G$ is rigidity-independent, then $\mathcal{V}(G)$ is Cohen-Macaulay.* □

*Proof.* If $E = \emptyset$, the result is trivial since $\mathcal{V}(G) \cong (\mathbb{P}^2)^{|V|}$. Otherwise, we add one edge at a time, applying Theorem 6.7 at each stage. □

## References


[1] C. De Concini and C. Procesi. Wonderful models of subspace arrangements. *Selecta Mathematica, New Series*, 1:459–494, 1995.

[2] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. Math.*, 139:183–225, 1994.

[3] J. Graver, B. Servatius, and H. Servatius. *Combinatorial Rigidity*. Amer. Math. Soc., 1993.

[4] C. Huneke. Strongly Cohen-Macaulay schemes and residual intersections. *Trans. Amer. Math. Soc.*, 277(2):739–763, 1983.

[5] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Eng. Math*, 4:331–340, 1970.

[6] P. Magyar. Borel-Weil theorem for configuration varieties and Schur modules. *Adv. Math.*, 134:328–366, 1998.

[7] J. L. Martin. *Graph Varieties*. PhD thesis, University of California, San Diego, 2002.

[8] A. Simis and W. V. Vasconcelos. The syzygies of the conormal module. *Amer. J. Math.*, 103:203–224, 1980.

[9] W. Whiteley. Some matroids from discrete applied geometry. *Contemp. Math.*, 197:171–311, 1996.