Math 824, Fall 2016
Problem Set #1

Instructions: Do all problems and typeset them in LaTeX. E-mail your final PDF file to Jeremy at `jlmartin@ku.edu` by **Tuesday, September 6, 11:59pm**.

You are encouraged to use the **header file** available from the course website.

Citations to the **lecture notes** refer to the version of August 30. Check that you have the most current version.

Exercise 1.1. Let n be a nonnegative integer.

(a) Prove that every interval in a Boolean algebra $\mathscr{B}_n$ is itself a Boolean algebra $\mathscr{B}_k$ for some $k \leq n$.
(b) Prove that every interval in the partition lattice Π_n is itself a partition lattice Π_k for some $k \leq n$.

Exercise 1.2. A **directed acyclic graph** or DAG, is a pair $G = (V, E)$, where V is a finite set of *vertices*; E is a finite set of *edges*, each of which is an ordered pair of distinct vertices; and E contains no directed cycles, i.e., no subsets of the form $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$ for any $v_1, \ldots, v_n \in V$.

(a) Let P be a poset with order relation $<$. Let $E = \{(v, w) \mid v, w \in P, \ v < w\}$. Prove that the pair (P, E) is a DAG.
(b) Let $G = (V, E)$ be a DAG. Define a relation $<$ on V by setting $v < w$ iff there is some directed path from v to w in G, i.e., iff E has a subset of the form $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$ with $v = v_1$ and $w = v_n$. Prove that this relation makes V into a poset.

(This problem is purely a technical exercise, but it does show that posets and DAGs are essentially the same thing.)

Exercise 1.3. Let n be a positive integer. Let D_n be the set of all positive-integer divisors of n (including n itself), partially ordered by divisibility.

(a) Prove that D_n is a ranked poset, and describe the rank function.
(b) For which values of n is D_n (i) a chain; (ii) a Boolean algebra? For which values of n, m is it the case that $D_n \cong D_m$?
(c) Prove that D_n is a *distributive lattice*, i.e., a lattice such that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in D_n$. Describe its meet and join operations and its join-irreducible elements.
(d) Prove that D_n is *self-dual*, i.e., there is a bijection $f : D_n \to D_n$ such that $f(x) \leq f(y)$ if and only if $x \geq y$.

Exercise 1.4. Let Δ be a simplicial complex on vertex set V, and let $v_0 \notin V$. The **cone over $\boldsymbol{\Delta}$** is the simplicial complex $C\Delta$ generated by all faces $\sigma \cup \{v_0\}$ for $\sigma \in \Delta$.

(a) Determine the f- and h-vectors of $C\Delta$ in terms of those of Δ.
(b) Show that Δ is shellable if and only if $C\Delta$ is shellable.

Exercise 1.5. Prove that conditions (1) and (2) in the definition of shellability (Defn. 1.19) are equivalent.

Exercise 1.6. Prove Proposition 1.20. (Hint: For the "Moreover" part, prove the contrapositive by showing that for all $i > 1$, there is some vertex $v \in R_i$ such that $R_i \setminus \{v\} = R_j$ for some $j < i$.)