Math 824 Problem Set #5
Due Wednesday, December 8

Problem #1 **(#1a)** Read the construction of the induced representation (§6.9 of the lecture notes). Start with an easy one: Determine the dimension of $\mathrm{Ind}_H^G(\rho)$ as a $\mathbb{C}$-vector space.

(#1b) Let n be a positive integer, and let D_n be the dihedral group $\langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1}\rangle$. Let C_n be the cyclic subgroup generated by x, and let ρ be the irreducible representation of C_n mapping x to $\zeta = e^{2\pi i/n} \in \mathbb{C}$. Show that $\mathrm{Ind}_{C_n}^{D_n}(\rho)$ is isomorphic to the "standard" two-dimensional representation of D_n as the group of symmetries of $\mathbb{R}^2$ fixing a regular n-gon).

Problem #2 **(#2a)** Work out the table of irreducible characters of $\mathfrak{S}_5$, by whatever means necessary. (You might want to enlist the services of a computer in doing the linear algebra.)

(#2b) Restrict all these characters to the alternating subgroup $\mathfrak{A}_5$, and work out as much as you can about its irreducible characters. Try assuming from the start that the conjugacy classes of $\mathfrak{A}_5$ are conjugacy classes in $\mathfrak{S}_5$. In fact this is false, as should become apparent in the course of the calculation.

(#2c) Let H be the subgroup of $\mathfrak{A}_5$ generated by a 5-cycle σ, and let χ be the 1-dimensional character defined by $\chi(\sigma) = \zeta$, where $\zeta = e^{2\pi i/5} \in \mathbb{C}$. With what you've already done, show that $\mathrm{Ind}_H^G \chi$ is an irreducible character of $\mathfrak{A}_5$.

Problem #3 Recall that for $\lambda, \mu \vdash n$, the Kostka number $K_{\lambda\mu}$ is defined as the number of column-strict tableaux of shape λ and content μ (that is, having μ_1 1's, μ_2 2's, etc.) Prove that $K_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$. (Together with the fact that $K_\lambda = 1$ for all λ, this implies that the Schur symmetric functions are a graded $\mathbb{Z}$-basis for Λ.)

Problem #4 Prove the second identity of Proposition 7.12 (in §7.3 of the lecture notes), namely

$$\prod_{i,j\geq 1}(1 + x_i y_j) \;=\; \sum_\lambda e_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}) \;=\; \sum_\lambda \varepsilon_\lambda \frac{p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y})}{z_\lambda}.$$

Hint: Mimic the proof of the *first* identity of Proposition 7.12.

Problem #5 **(#5a)** For $w \in \mathfrak{S}_n$, let $(P(w), Q(w))$ be the pair of tableaux produced by the RSK algorithm from w. Denote by w^* the reversal of w in one-line notation (for instance, if $w = 57214836$, as in Example 7.16 from the lecture notes, then $w^* = 63841275$). Prove that $P(w^*) = P(w)^T$ (where ^{T} means transpose). Hint: Figure out how to describe the rows and columns of $P(w)$ in terms of subsequences of w.

(#5b) *Open problem:* For which permutations does $Q(w^*) = Q(w)$? Maple computation indicates that the number of such permutations is

$$\begin{cases} \dfrac{2^{(n-1)/2}(n-1)!}{((n-1)/2)^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

but I don't know a combinatorial (or even an algebraic) reason.

(#5c) *Open problem:* For which permutations does $Q(w^*) = Q(w)^T$? I have no idea what the answer is. The sequence $(q_1, q_2, \dots) = (1, 2, 2, 12, 24, 136, 344, 2872, 7108, \dots)$, where $q_n = \#\{w \in \mathfrak{S}_n \mid Q(w^*) = Q(w)^T\}$, does not seem to appear in the Online Encyclopedia of Integer Sequences.