Math 824, Fall 2010
Problem Set #3
Due date: Friday 10/8/10

Problem #1 Let X and Y be disjoint sets of vertices, and let B be an X,Y-bipartite graph: that is, every edge of B has one endpoint in each of X and Y. For $V = \{x_1, \dots, x_n\} \subset X$, a *transversal* of V is a set $W = \{y_1, \dots, y_n\} \subset Y$ such that $x_i y_i$ is an edge of B. Let $\mathscr{I}$ be the family of all subsets of X that have a transversal. Prove that $\mathscr{I}$ is a matroid independence system. (A matroid that arises in this way is called a *transversal matroid*.)

Problem #2 Let M be a matroid on ground set E with rank function r.

(#2a) Let M^* be the dual matroid to M, and let r^* be its rank function. Find a formula for r^* in terms of r.

(#2b) Use the formula from part (a), together with the corank-nullity form of the Tutte polynomial, to prove that $T(M, x, y) = T(M^*, y, x)$ for every matroid M.

Problem #3 Let $G = (V, E)$ be a graph with n vertices and c components. For a vertex coloring $f : V \to \mathbb{N}$, let $i(f)$ denote the number of "improper" edges, i.e., whose endpoints are assigned the same color. The *(Crapo) coboundary polynomial* of G is

$$\bar{\chi}_G(q; t) \;=\; q^{-1} \sum_{f : V \to [q]} t^{i(f)}.$$

This is evidently a stronger invariant than the chromatic polynomial of G, which can be obtained as $q\bar{\chi}_G(q, 0)$. In fact, the coboundary polynomial provides the same information as the Tutte polynomial.

(#3a) Prove that

$$\bar{\chi}_G(q; t) \;=\; (t-1)^{n-c} T_G\left(\frac{q+t-1}{t-1}, t\right).$$

(#3b) Express the Tutte polynomial in terms of the coboundary polynomial.

Problem #4 Let P be a chain-finite poset. The *kappa function* of P is the element of the incidence algebra $I(P)$ defined by $\kappa(x, y) = 1$ if $x \lessdot y$, $\kappa(x, y) = 0$ otherwise.

(#4a) Give a condition on κ that is equivalent to P being ranked.

(#4b) Give combinatorial interpretations of $\kappa * \zeta$ and $\zeta * \kappa$.

Problem #5 Let Π_n be the lattice of set partitions of $[n]$. Recall that the order relation on Π_n is given as follows: if $\pi, \sigma \in \Pi_n$, then $\pi \leq \sigma$ if every block of π is contained in some block of σ (for short, "π refines σ"). In this problem, you're going to calculate the number $\mu_n := \mu_{\Pi_n}(\hat{0}, \hat{1})$.

(#5a) Calculate μ_n by brute force for $n = 1, 2, 3, 4$. Make a conjecture about the value of μ_n in general.

(#5b) Define a function $f : \Pi_n \to \mathbb{Q}[x]$ as follows: if X is a finite set of cardinality x, then

$$f(\pi) = \#\big\{h : [n] \to X \quad \big| \quad h(s) = h(s') \iff s, s' \text{ belong to the same block of } \pi\big\}.$$

For example, if $\pi = \hat{1} = \{\{1, 2, \dots, n\}\}$ is the one-block partition, then $f(\pi)$ counts the constant functions from $[n]$ to X, so $f(\pi) = x$. Find a formula for $f(\pi)$ in general.

(#5c) Let $g(\pi) = \sum_{\sigma \geq \pi} f(\sigma)$. Prove that $g(\pi) = x^{|\pi|}$ for all $\pi \in \Pi_n$. (Hint: What kinds of functions are counted by the sum?)

(#5d) Apply Möbius inversion and an appropriate substitution for x to calculate μ_n.

Problem #6 [Optional; requires a bit of abstract algebra.] Let n be a positive integer, and let ζ be a primitive n^{th} root of unity. The *cyclotomic matroid* Y_n is represented over $\mathbb{Q}$ by the numbers $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, regarded as elements of the cyclotomic field extension $\mathbb{Q}(\zeta)$. Thus, the rank of Y_n is the dimension of $\mathbb{Q}(\zeta)$ as a $\mathbb{Q}$-vector space, which is given by the Euler phi function.

Prove as many of the following facts as you want:

(1) if n is prime, then $Y_n \cong U_{n-1}(n)$;
(2) if m is the square-free part of m (i.e., the product of all the primes dividing n — e.g., the square-free part of 56 is 14) then Y_n is the direct sum of n/m copies of Y_m;
(3) if $n = pq$, where p, q are distinct primes, then $Y_n \cong M(K_{p,q})^*$ — that is, the dual of the graphic matroid of the complete bipartite graph $K_{p,q}$.