Math 821, Spring 2018
Problem Set #2
Deadline: Friday, February 16, 5:00pm

Instructions: Typeset your solutions in LaTeX. You are encouraged to use the Math 821 header file. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name and the problem set number (e.g., `Riemann2.pdf`). Collaboration is encouraged, but each student must write up his or her solutions independently and acknowledge all collaborators.

Problem #1 (Hatcher, p.20, #22) Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge line ℓ of P in a subset of its vertices. Describe the homotopy type of the "surface of revolution" Z obtained by rotating X about ℓ. (Examples: If X has two vertices and one edge, then Z is a 2-sphere, a disk or a cylinder according as X has two, one or zero vertices on ℓ. If X has one vertex not on ℓ with a loop attached, then Z is a torus.)

Problem #2 Recall that for a space X and basepoint $p \in X$, we have defined $\pi_1(X,p)$ to be the set of homotopy classes of p-loops on X, or equivalently of continuous functions $S^1 \to X$. Recall also that S^0 consists of two points (let's call them a and b) with the discrete topology. Accordingly, we could define $\pi_0(X,p)$ to be the set of homotopy classes of continuous functions $f : S^0 \to X$ such that $f(a) = p$. Describe the set $\pi_0(X,p)$ intrinsically in terms of X. Is there a natural group structure on $\pi_0(X,p)$? (Of course every set can be made into a group somehow, so the question is whether you can do it in a way depends only on, and tell you something about, the topology of X.)

Problem #3 (Hatcher, p.38, #2) Show that the change-of-basepoint homomorphism β_h (see p.28) depends only on the homotopy class of the path h.

Problem #4 (Hatcher, p.38, #7) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on *one* of the boundary circles, but not by any homotopy f_t that is stationary on *both* boundary circles.

Problem #5 [Hatcher p.38 #8] Does the Borsuk-Ulam theorem hold for the torus? In other words, for every continuous $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x,y) \in S^1 \times S^1$ such that $f(x,y) = f(-x,-y)$? Why or why not?

Problem #6 [Hatcher p.39 #12] Fix $p \in S^1$. Show that every homomorphism $\pi_1(S^1,p) \to \pi(S^1,p)$ can be realized as the induced homomorphism ϕ_* for some continuous $\phi : S^1 \to S^1$.

Problem #7 [Hatcher, p.52, #1] Recall that the **center** of a group G is defined as $Z(G) = \{g \in G : gh = hg \ \forall h \in G\}$.

(#7a) Show that the free product $G * H$ of nontrivial groups G and H has trivial center.

(#7b) Show that the only elements of $G * H$ of finite order are the conjugates of finite-order elements in $G \cup H$.