Math 821, Spring 2018
Problem Set #1
Deadline: Friday, February 2, 5:00pm

Instructions: Typeset your solutions in LaTeX. You are encouraged to use the Math 821 header file. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name and the problem set number (e.g., `Euler1.pdf`). Collaboration is encouraged, but each student must write up his or her solutions independently and acknowledge all collaborators.

As always, "space" means "topological space" and "map" means "continuous function".

Problem #1 (a) Let X be a path-connected space. Prove that X is connected. (Recall that the converse is not true — the topologists' sine curve is a counterexample.)

(b) Prove that if X is connected and every point has a path-connected neighborhood, then X is path-connected. (The hypothesis does hold for cell complexes.)

Problem #2 Let X and Y be spaces and let $f : X \twoheadrightarrow Y$ be a map that is onto. Prove that if X is compact, then so is Y. (This should not be hard.)

These first two problems were intended as review of general topology; now onto algebraic topology!

Problem #3 [Hatcher, Chapter 0, #3, more or less] (a) Show that the relation "X is homotopy equivalent to Y" is an equivalence relation.

(b) Fix spaces X, Y and let f, g be maps $X \to Y$. Show that the relation "f is homotopic to g" is an equivalence relation.

(c) Let $f : X \to Y$ be a homotopy equivalence. Show that any map $g : X \to Y$ homotopic to f is a homotopy equivalence.

Problem #4 [Hatcher, Chapter 0, #1] Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point.

Problem #5 [Hatcher, Chapter 0, #14] Given nonnegative integers v, e, f with $v > 0$, $f > 0$, and $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells. (Do not use any facts you may happen to know about spanning trees or Euler characteristic.)

Problem #6 [Hatcher, Chapter 0, #21, expanded] Let X be a connected space that is the union of a finite number of 2-spheres, any two of which intersect in at most one point. Show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

More precisely, let $G = (V, E)$ be the graph whose vertices correspond to the 2-spheres comprising X, with two vertices adjacent if the corresponding spheres intersect. You should be able to show that

$$X \simeq (S^1)^{|E|-|V|+1} \vee (S^2)^{|V|}.$$