Math 821, Spring 2014
Problem Set #7
Due date: Friday, May 2

> **Note:** In all cases, "compute the homology groups" means "compute $H_n(X)$ for $n > 0$" — you don't have to incessantly repeat that $H_0(X) = \mathbb{Z}$ for path-connected spaces.

Problem #1 [Hatcher p.156 #9abc] Compute the homology groups of the following spaces:

(a) The quotient of $\mathbb{S}^2$ obtained by identifying the north and south poles to a point.

(b) $\mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$.

(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks, and then identifying all three resulting circles together via homomorphisms preserving clockwise orientations of these circles.

Problem #2 [Hatcher p.157 #19] Compute $H_i(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{R}P^n$ with $\mathbb{R}P^m$ as its m-skeleton.

Problem #3 [Hatcher p.156 #11] Let K be the 3-dimensional cell complex obtained from the cube I^3 by identifying each pair of opposite faces via a one-quarter twist. (See exercise #14 on p.54.) Compute the homology groups $\tilde{H}_n(K;\mathbb{Z})$ and $\tilde{H}_n(K;\mathbb{Z}_2)$ for $n > 0$.

Problem #4 [Hatcher p.157 #20,22] In this problem χ denotes Euler characteristic.

(a) Let X, Y be finite CW-complexes. Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

(b) Let X be a finite CW complex and let $\tilde{X} \xrightarrow{p} X$ be an n-sheeted covering space. Show that $\chi(\tilde{X}) = n \cdot \chi(X)$.

Problem #5 [Hatcher p.155 #2, modified] Given a map $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$, show that there is some point $x \in \mathbb{S}^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Hint: Use the fact that $\mathbb{S}^{2n}$ is a covering space of $\mathbb{R}P^{2n}$.)

Problem #6 [Hatcher p.157 #28(a), modified] (a) Use a Mayer-Vietoris sequence to compute the homology groups of the space X obtained from a torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ by attaching a Möbius band M via a homeomorphism from the boundary circle C of M to the circle $\mathbb{S}^1 \times \{x_0\}$ in the torus.

(b) How does the answer change if C is attached to a closed loop that wraps k times around the first circle (i.e., via the path $f : I \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $f(t) = (e^{2\pi ikt}, e^{2\pi it})$)?

Problem #7 [Hatcher p.158 #29] The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a compact closed 3-manifold X. Compute the homology groups of X using a Mayer-Vietoris sequence. Also compute the relative groups $H_i(R, M_g)$.