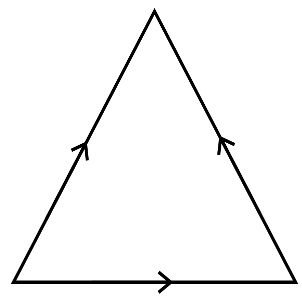
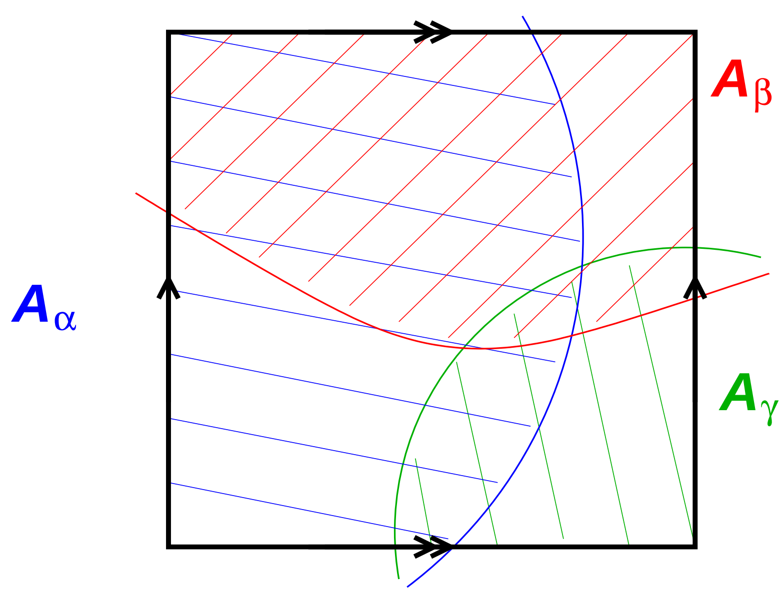
Math 821 Problem Set #4
Posted: Friday 3/11/11
Due date: Monday 3/28/11

Problem #1 (Hatcher, p.52, #1) Show that the free product $G * H$ of nontrivial groups G, H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Problem #2 The *dunce hat* is the space D obtained from a triangle by identifying all three edges with each other, with the orientations indicated below. Give two separate proofs that D is simply-connected. (There are at least three: (a) show that D is in fact contractible; (b) use Van Kampen's theorem; (c) a slick one-line proof using something we did in class.)



Problem #3 Consider the standard picture of the torus $T = S^1 \times S^1$ as a quotient space of the square. Why does the decomposition $T = A_\alpha \cup A_\beta \cup A_\gamma$ shown below, together with Van Kampen's theorem, *not* imply that T is simply-connected?



(Continued on back.)

Problem #4 (Hatcher, p.53, #4, modified) Let $n \geq 1$ be an integer, and let $X \subset \mathbb{R}^3$ be the union of n distinct rays emanating from the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

(Note: This problem is the tip of an iceberg: the theory of *subspace arrangements.* There are many beautiful theorems about the topology of such things, often using tools from areas such as algebraic combinatorics and the theory of Coxeter groups.)

Problem #5 Let $a_1, \ldots, a_n$ be nonzero integers. Construct a cell complex X from S^1 as follows: For each $j = 1, \ldots, n$, attach a 2-cell to S^1 by wrapping it around the circle a_j times. Compute $\pi_1(X)$.

Problem #6 (Hatcher, p.53, #6, modified) Let X be a path-connected cell complex, and let Y be a cell complex obtained from X by attaching an n-cell for some $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.