Math 821 Problem Set #2
Posted: Friday 2/11/11
Due date: Monday 2/21/11

Problem #1 (Hatcher, p.18, #2) Construct an explicit deformation retraction of $X = \mathbb{R}^n \setminus \{0\}$ onto S^{n-1}. ("Explicit" means that you should write down an actual formula for the map $f_t : X \to X$, and check that the family of maps you have defined satisfies the conditions of a deformation retraction.)

Problem #2 (Hatcher, p.19, #12) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

Problem #3 (Hatcher, p.19, #17) Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts. (This implies that these two spaces are homotopy-equivalent.)

Problem #4 Let X, Y be cell complexes of finite type. Recall that the *f-polynomial* of X is

$$f(X;q) = \sum_{\text{cells } e^i_\alpha \in X} q^i = \sum_{i=0}^{\dim X} f_i q^i$$

where f_i is the number of i-dimensional cells (and recall that "of finite type" means that $f_i < \infty$ for each i). In terms of $f(X;q)$ and $f(Y;q)$, find formulas for

(a) $f(X \times Y; q)$;
(b) $f(X/Y; q)$ (assuming that (X,Y) is a CW-pair);
(c) $f(CX; q)$;
(d) $f(X * Y; q)$.

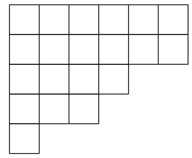
You don't have to give detailed proofs. (See pp. 8–10 of Hatcher for the definitions of these operations.)

Problem #5 Let $0 \leq k \leq n$. Recall from class that the *Grassmannian* $G(k,n)$ is defined as the space of k-dimensional subspaces $V \subset \mathbb{R}^n$, so that in particular, $G(1, \mathbb{R}^n) = \mathbb{R}P^{n-1}$. (Fact: Everything in this problem works the same way if you change $\mathbb{R}$ to $\mathbb{C}$, except that the dimensions of all the cells get doubled.)

(#5a) Work out an explicit cell decomposition for $G(2,4)$ as a finite CW-complex. That is, describe how to decompose the set $G(2,4)$ into pieces, each of which is isomorphic to a $\mathbb{R}$-vector space. If you do this correctly (hint: row-reduced echelon form), then the isomorphisms should be straightforward from the construction.

(#5b) Describe the attaching poset of $G(2,4)$. (Recall that this is the partially ordered set whose elements are the cells e_α, and whose order relation is given by $e_\alpha \geq e_\beta$ if $\overline{e_\alpha} \supseteq e_\beta$).

(#5c) A *Ferrers diagram* is a collection of square boxes that are top- and right-justified: for instance,



Write out the poset $P(2,2)$ of all Ferrers diagrams with at most two rows and at most two columns, ordered by containment (as sets of squares). Compare it to your previous answer.

(#5d) Make as many conjectures as you dare about the cell structure of $G(k,n)$. (In particular, what happens to the attaching poset if you reverse all the relations?)

Problem #6 (Extra credit; Hatcher, p.19, #16) Show that S^∞ is contractible.