Friday 5/2

The Frobenius Characteristic

Let R be a ring. Denote by $C\ell_R(\mathfrak{S}_n)$ the vector space of R-valued class functions on the symmetric group $\mathfrak{S}_n$. If no R is specified, we assume $R = \mathbb{C}$. Define

$$C\ell(\mathfrak{S}) \;=\; \bigoplus_{n\geq 0} C\ell(\mathfrak{S}_n).$$

We make $C\ell(\mathfrak{S})$ into a graded ring as follows. For $f_1 \in C\ell(\mathfrak{S}_{n_1})$ and $f_2 \in C\ell(\mathfrak{S}_{n_2})$, we can define a function

$$f_1 \otimes f_2 \in C\ell(\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$$

by

$$f_1 \otimes f_2(w_1, w_2) = f_1(w_1) f_2(w_2).$$

There is a natural inclusion of groups $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}) \hookrightarrow \mathfrak{S}_{n_1+n_2}$, so we can define $f_1 \cdot f_2 \in C\ell(\mathfrak{S}_{n_1+n_2})$ by means of the induced "character":

$$f_1 \cdot f_2 = \operatorname{Ind}_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}} (f_1 \otimes f_2)$$

(since the formula for induced characters can be applied to arbitrary class functions).

This product makes $C\ell(\mathfrak{S})$ into a graded $\mathbb{C}$-algebra. (We won't prove this.)

For a partition $\lambda \vdash n$, let 1_λ be the indicator function on the conjugacy class $C_\lambda \subset \mathfrak{S}_n$, and let

$$\mathfrak{S}_\lambda \;=\; \mathfrak{S}_{\{1,\dots,\lambda_1\}} \;\times\; \mathfrak{S}_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \;\times\; \cdots \;\times\; \mathfrak{S}_{\{n-\lambda_\ell+1,\dots,n\}} \;\subset\; \mathfrak{S}_n.$$

For $w \in \mathfrak{S}_n$, denote by $\lambda(w)$ the cycle-shape of w, expressed as a partition.

Definition 1. The **Frobenius characteristic** is the map

$$\mathbf{ch} : \; C\ell_{\mathbb{C}}(\mathfrak{S}) \to \Lambda_{\mathbb{C}}$$

defined on $f \in C\ell(\mathfrak{S}_n)$ by

$$\mathbf{ch}(f) = \frac{1}{n!} \sum_{w\in\mathfrak{S}_n} \overline{f(w)} p_{\lambda(w)}.$$

Equivalently,

$$\mathbf{ch}(f) = \langle f, \psi \rangle_{\mathfrak{S}_n}$$

where ψ is the class function $\mathfrak{S}_n \to \Lambda^n$ defined by

(1) $$\psi(w) = p_{\lambda(w)}.$$

Theorem 1.

1. ***ch*** *is a ring isomorphism.*
2. ***ch*** *is an isometry, i.e., it preserves inner products:*
$$\langle f, g \rangle_{\mathfrak{S}_n} \;=\; \langle \mathbf{ch}(f), \mathbf{ch}(g) \rangle_\Lambda.$$
3. ***ch*** *restricts to an isomorphism* $C\ell_{\mathbb{Z}}(\mathfrak{S}) \to \Lambda_{\mathbb{Z}}$.
4. $1_\lambda \;\mapsto\; p_\lambda / z_\lambda$.
5. $\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \chi_{\text{triv}} \;\mapsto\; h_\lambda$.
6. $\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \chi_{\text{sign}} \;\mapsto\; e_\lambda$.
7. *The irreducible characters of* $\mathfrak{S}_n$ *are the* $\mathbf{ch}^{-1}(s_\lambda)$.
8. *For all characters* χ, *we have* $\mathbf{ch}(\chi \otimes \chi_{\text{triv}}) = \omega(\mathbf{ch}(\chi))$.

I'll prove a few of these assertions. Recall that

$$|C_\lambda| = n!/z_\lambda. \tag{2}$$

Therefore

$$\mathbf{ch}(1_\lambda) \;=\; \frac{1}{n!}\sum_{w\in C_\lambda} p_\lambda \;=\; p_\lambda/z_\lambda$$

which proves assertion (4). It follows that **ch** is (at least) a graded $\mathbb{C}$-vector space isomorphism (since $\{1_\lambda\}$ and $\{p_\lambda/z_\lambda\}$ are graded $\mathbb{C}$-bases for $C\ell(\mathfrak{S})$ and Λ respectively).

To show assertion (2), it suffices to check it on these bases. Let $\lambda,\mu\vdash n$; then

$$\langle 1_\lambda,\ 1_\mu\rangle_{\mathfrak{S}_n} \;=\; \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\overline{1_\lambda(w)}1_\mu(w) \;=\; \frac{1}{n!}|C_\lambda|\delta_{\lambda\mu} \;=\; \delta_{\lambda\mu}/z_\lambda,$$

$$\left\langle \frac{p_\lambda}{z_\lambda},\ \frac{p_\mu}{z_\mu}\right\rangle_\Lambda \;=\; \frac{1}{\sqrt{z_\lambda z_\mu}}\left\langle \frac{p_\lambda}{\sqrt{z_\lambda}},\ \frac{p_\mu}{\sqrt{z_\mu}}\right\rangle_\Lambda \;=\; \frac{1}{\sqrt{z_\lambda z_\mu}}\delta_{\lambda\mu} \;=\; \delta_{\lambda\mu}/z_\lambda.$$

Next we check that **ch** is a ring homomorphism (hence an isomorphism). Let $f\in\mathfrak{S}_j$, $g\in\mathfrak{S}_k$, and $n=j+k$. Then

$$\mathbf{ch}(f\cdot g) \;=\; \left\langle \mathrm{Ind}_{\mathfrak{S}_j\times\mathfrak{S}_k}^{\mathfrak{S}_n}(f\otimes g),\ \psi\right\rangle_{\mathfrak{S}_n}$$

(where ψ is as in (1))

$$=\; \left\langle f\otimes g,\ \mathrm{Res}_{\mathfrak{S}_j\times\mathfrak{S}_k}^{\mathfrak{S}_n}\,\psi\right\rangle_{\mathfrak{S}_j\times\mathfrak{S}_k}$$

(by Frobenius reciprocity)

$$\begin{aligned}
&=\; \frac{1}{j!\,k!}\sum_{(w,x)\in\mathfrak{S}_j\times\mathfrak{S}_k}\overline{f\otimes g(w,x)}\;p_{\lambda(w,x)}\\
&=\; \left(\frac{1}{j!}\sum_{w\in\mathfrak{S}_j}\overline{f(w)}\;p_{\lambda(w)}\right)\left(\frac{1}{k!}\sum_{x\in\mathfrak{S}_k}\overline{g(x)}\;p_{\lambda(x)}\right)\\
&=\; \mathbf{ch}(f)\ \mathbf{ch}(g).
\end{aligned}$$

More Fundamental Results

1. The Murnaghan-Nakayama Rule.

We now know that the irreducible characters of $\mathfrak{S}_n$ are $\chi^\lambda = \mathbf{ch}^{-1}(s_\lambda)$ for $\lambda\vdash n$. The Murnaghan-Nakayama Rule gives a formula for the value of the character χ^λ on the conjugacy class C_μ in terms of *rim-hook tableaux*. Here is an example of a rim-hook tableau of shape $\lambda=(5,4,3,3,1)$ and content $\mu=(6,3,3,2,1,1)$:

1	1	1	4	4
1	2	3	3	
1	2	3		
1	2	6		
5				

Note that the columns and row are weakly increasing, and for each i, the set $H_i(T)$ of cells containing an i is contiguous.

Theorem 2 (Murnaghan-Nakayama Rule (1937))**.**

$$\chi^\lambda(C_\mu) = \sum_{\substack{\text{rim-hook tableaux } T \\ \text{of shape } \lambda \text{ and content } \mu}} \prod_{i=1}^{n} (-1)^{1+\mathrm{ht(H_i(T))}}.$$

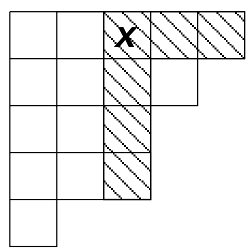
For example, the heights of $H_1, \ldots, H_6$ in the rim-hook tableau above are 4, 3, 2, 1, 1, 1. There are an even number of even heights, so this rim-hook tableau contributes 1 to $\chi\lambda(C_\mu)$.

An important special case is when $\mu = (1, 1, \ldots, 1)$, i.e., since then $\chi^\lambda(C_\mu) = \chi^\lambda(1_{\mathfrak{S}_n})$ i.e., the dimension of the irreducible representation S^λ of $\mathfrak{S}_n$ indexed by λ. On the other hand, a rim-hook tableau of content μ is just a standard tableau. So the Murnaghan-Nakayama Rule implies the following:

Corollary 3. $\dim S^\lambda = f^\lambda$.

This begs the question of how to calculate f^λ (which you may have been wondering anyway). There is a beautiful formula in terms of *hooks*.

For each cell x in the Ferrers diagram of λ, let $h(x)$ denote its *hook length*: the number of cells due east of, due south of, or equal to x. In the following example, $h(x) = 6$.



Theorem 4 (Hook Formula of Frame, Robinson, and Thrall (1954))**.** *Let* $\lambda \vdash n$. *Then*

$$f^\lambda \;=\; \frac{n!}{\prod_{x\in\lambda} h(x)}.$$

Example 1. For $\lambda = (5, 4, 3, 3, 1) \vdash 16$ as above, here are the hook lengths:

9	7	6	3	1
7	5	4	1	
5	3	2		
4	2	1		
1				

Therefore

$$f^\lambda \;=\; \frac{14!}{9\cdot 7^2\cdot 6\cdot 5^2\cdot 4^2\cdot 3^2\cdot 2^2\cdot 1^4} \;=\; 2288.$$

Example 2. For $\lambda = (n, n) \vdash 2n$, the hook lengths are

$n + 1,\ n,\ n - 1,\ \ldots,\ 2$ (top row),

$n,\ n - 1,\ n - 2,\ \ldots,\ 1$ (bottom row).

Therefore

$$f^\lambda \;=\; \frac{(2n)!}{(n+1)!\; n!} \;=\; \frac{1}{n+1}\binom{2n}{n}$$

which is the n^{th} Catalan number (as we already know).