Monday 4/21

Restricted and Induced Representations

Definition 1. Let $H \subset G$ be finite groups, and let $\rho : G \to GL(V)$ be a representation of G. Then the restriction of ρ to H is a representation of G, denoted $\mathrm{Res}^G_H(\rho)$. Likewise, the restriction of $\chi = \chi_\rho$ to H is a chartacter of H denoted by $\mathrm{Res}^G_H(\chi)$.

Notice that restricting a representation does not change its character. OTOH, whether or not a representation is irreducible can change upon restriction.

Example 1. Let C_λ denote the conjugacy class in $\mathfrak{S}_n$ of permutations of cycle-shape λ. Recall that $G = \mathfrak{S}_3$ has an irrep whose character $\psi = \chi_\rho$ is given by

$$\psi(C_{111}) = 2, \qquad \psi(C_{21}) = 0, \qquad \psi(C_3) = -1.$$

Let $H = \mathfrak{A}_3 \subseteq \mathfrak{S}_3$. This is an abelian group (isomorphic to $\mathbb{Z}/3\mathbb{Z}$)), so the two-dimensional representation $\mathrm{Res}^G_H(\rho)$ is not irreducible. Specifically, let $\omega = e^{2\pi i/3}$ The table of irreducible characters of $\mathfrak{A}_3$ is as follows:

	1_G	$(1\,2\,3)$	$(1\,3\,2)$
χ_{triv}	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

Now it is evident that $\mathrm{Res}^G_H \psi = [2, -1, -1] = \chi_1 + \chi_2$. Note, by the way, that the conjugacy class $C_3 \subset \mathfrak{S}_3$ splits into two singleton conjugacy classes in $\mathfrak{A}_3$, a common phenomenon when working with restrictions.

Next, we construct a representation of G from a representation of a subgroup $H \subset G$.

Definition 2. Let $H \subset G$ be finite groups, and let $\rho : H \to GL(W)$ be a representation of H. Define the *induced representation* $\mathrm{Ind}^G_H(\rho)$ as follows.

- Choose a set of left coset representatives $B = \{b_1, \dots, b_r\}$ for H in G. That is, every $g \in G$ can be expressed uniquely as $g = b_j h$, for some $b_j \in B$ and $h \in H$.
- Let $\mathbb{C}[G/H]$ be the $\mathbb{C}$-vector space with basis B.
- Let $V = \mathbb{C}[G/H] \otimes_{\mathbb{C}} W$.
- Let $g \in G$ act on $b_i \otimes w \in V$ as follows. Find the unique $b_i \in B$ and $h \in H$ such that $gb_i = b_j h$, and put
$$g \cdot (b_i \otimes w) = b_j \otimes hw.$$
This makes more sense if we observe that $g = b_j h b_i^{-1}$, so that the equation becomes
$$b_j h b_i^{-1} \cdot (b_i \otimes w) = b_j \otimes hw.$$
- Extend this to a representation of G on V by linearity.

Proposition 1. *$\mathrm{Ind}^G_H(\rho)$ is a representation of G that is independent of the choice of B. Moreover, for all $g \in G$,*

$$\chi_{\mathrm{Ind}^G_H(\rho)}(g) = \frac{1}{|H|} \sum_{\substack{k \in G \\ k^{-1}gk \in H}} \chi_\rho(k^{-1}gk).$$

Proof. First, we verify that $\mathrm{Ind}^G_H(\rho)$ is a representation. Let $g, g' \in G$ and $b_i \otimes w \in V$. Then there is a unique $b_j \in B$ and $h \in H$ such that

$$gb_i = b_j h \tag{1}$$

and in turn there is a unique $b_\ell \in B$ and $h' \in H$ such that

$$g' b_j = b_\ell h'. \tag{2}$$

We need to verify that

$$g' \cdot (g \cdot (b_i \otimes w)) \;=\; (g'g) \cdot (b_i \otimes w). \tag{3}$$

Indeed,

$$\Big(g' \cdot (g \cdot (b_i \otimes w)) \;=\; g' \cdot (b_j \otimes hw)\Big) \;=\; b_\ell \otimes h'hw.$$

On the other hand, by (1) and (2), $gb_i = b_j h b_i^{-1}$ and $g' = b_\ell h' b_j^{-1}$, so

$$(g'g) \cdot (b_i \otimes w) \;=\; (b_\ell h' h b_i^{-1}) \cdot (b_i \otimes w) \;=\; b_\ell \otimes h'hw$$

as desired.

Now that we know that $\mathrm{Ind}_H^G(\rho)$ is a representation of G on V, we find its character on an arbitrary element $g \in G$. Regard $\mathrm{Ind}_H^G(\rho)(g)$ as a block matrix with r row and column blocks, each of size $\dim W$ and corresponding to the subspace of V of vectors of the form $b_i \otimes w$ for some fixed b_i. The block in position (i, j) is

- a copy of $\rho(h)$, if $gb_i = b_j h$ for some $h \in H$,
- zero otherwise.

Therefore,

$$\begin{aligned}
\chi_{\mathrm{Ind}_H^G(\rho)}(g) &= \mathrm{tr}\,(g:\ \mathbb{C}[G/H] \otimes_{\mathbb{C}} W \to \mathbb{C}[G/H] \otimes_{\mathbb{C}} W) \\
&= \sum_{\substack{i \in [r]: \\ gb_i = b_i h \\ (\exists h \in H)}} \chi_\rho(h) \\
&= \sum_{\substack{i \in [r]: \\ b_i^{-1} g b_i \in H)}} \chi_\rho(b_i^{-1} g b_i) \\
&= \frac{1}{|H|} \sum_{\substack{i \in [r]: \\ b_i^{-1} g b_i \in H}} \sum_{h \in H} \chi_\rho(h^{-1} b_i^{-1} g b_i h) \\
&= \frac{1}{|H|} \sum_{\substack{k \in G: \\ k^{-1} g k \in H}} \chi_\rho(k^{-1} g k).
\end{aligned}$$

Here we have put $k = b_i h$, which runs over all elements of G. The character of $\mathrm{Ind}_H^G(\rho)$ is independent of the choice of B; therefore, so is the representation itself. □

Theorem 2 (Frobenius Reciprocity). *Let $H \subset G$ be finite groups. Let χ be a chararacter of H and let ψ be a character of G. Then*

$$\Big\langle \mathrm{Ind}_H^G \chi,\ \psi \Big\rangle_G = \big\langle \chi,\ \mathrm{Res}_H^G \psi \big\rangle_H .$$

Example 2. Sometimes, Frobenius reciprocity suffices to calculate the isomorphism type of an induced representation. Let ψ, χ_1 and χ_2 be as in Example 1. We would like to compute $\mathrm{Ind}_H^G \chi_1$. By Frobenius reciprocity

$$\Big\langle \mathrm{Ind}_H^G \chi_1,\ \psi \Big\rangle_G = \big\langle \chi_1,\ \mathrm{Res}_H^G \psi \big\rangle_H = 1.$$

But ψ is irreducible. Therefore, it must be the case that $\mathrm{Ind}_H^G \chi_1 = \psi$, and the corresponding representations are isomorphic. The same is true if we replace χ_1 with χ_2.

Proof next time.