We worked out the irreducible characters of $S_4$ ad hoc. We’d like to have a way of calculating them in general.

Recall that a partition of $n$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weakly decreasing positive integers whose sum is $n$. We write $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$. The number of partitions of $n$ is denoted $p(n)$.

For $\lambda \vdash n$, let $C_\lambda$ be the conjugacy class in $S_n$ consisting of all permutations with cycle shape $\lambda$. Since the conjugacy classes are in bijection with the partitions of $n$, it makes sense to look for a set of representations indexed by partitions.

**Definition 1.** Let $\mu = (\mu_1, \ldots, \mu_m) \vdash n$. The Ferrers diagram of shape $\mu$ is the top- and left-justified array of boxes with $\mu_i$ boxes in the $i^{th}$ row. A Young tableau of shape $\mu$ is a Ferrers diagram with the numbers $1, 2, \ldots, n$ placed in the boxes, one number to a box. Two tableaux $T, T'$ of shape $\mu$ are row-equivalent, written $T \sim T'$, if the numbers in each row of $T$ are the same as the numbers in the corresponding row of $T'$. A tabloid of shape $\mu$ is an equivalence class of tableaux under row-equivalence. A tabloid can be represented as a tableau without vertical lines separating numbers in the same row. We write $\sh(T) = \mu$ to indicate that a tableau or tabloid $T$ is of shape $\mu$.

\[
\begin{array}{c|c|c}
& 1 & 3 & 6 \\
\hline
1 & 2 & 7 \\
2 & 7 \\
4 & 5 \\
\end{array}
\quad
\begin{array}{c|c|c}
& 1 & 3 & 6 \\
\hline
2 & 7 \\
4 & 5 \\
\end{array}
\]

Ferrers diagram Young tableau Young tabloid

A Young tabloid can be regarded as a set partition $(T_1, \ldots, T_m)$ of $[n]$, in which $|T_i| = \mu_i$. The order of the blocks $T_i$ matters, but not the order of digits within each block. Thus the number of tabloids of shape $\mu$ is

\[
\left(\begin{array}{c}
n \\
\mu
\end{array}\right) = \frac{n!}{\mu_1! \cdots \mu_m!}.
\]

The symmetric group $S_n$ acts on tabloids by permuting the numbers. Accordingly, we have a permutation representation $(\rho_\mu, V^\mu)$ of $S_n$ on the vector space $V^\mu$ of all $\mathbb{C}$-linear combinations of tabloids of shape $\mu$.

**Example 1.** For $n = 3$, the characters of the representations $\rho_\mu$ are as follows.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>111</th>
<th>21</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = (3)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\mu = (2,1)$</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\mu = (1,1,1)$</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$</td>
<td>C_\lambda</td>
<td>$</td>
<td>1</td>
</tr>
</tbody>
</table>

(1)

Many familiar representations of $S_n$ can be expressed in this form.

- There is a unique tabloid of shape $\mu = (n)$: $T = \begin{array}{c}1 & 2 & \cdots & n\end{array}$. Every permutation fixes $T$, so $\rho_{(n)} \cong \rho_{\text{triv}}$. 
• The tabloids of shape $\mu = (1, 1, \ldots, 1)$ are just the permutations of $[n]$. Therefore
  \[ \rho_{(1, 1, \ldots, 1)} \cong \rho_{\text{reg}}. \]
• A tabloid of shape $\mu = (n-1, 1)$ is determined by its singleton part. So the representation $\rho_\mu$ is isomorphic to the action of $S_n$ on this part by permutation; that is
  \[ \rho_{(n-1, 1)} \cong \rho_{\text{def}}. \]

For $n = 3$, the table in $\Box$ is triangular, which implies immediately that the characters $\rho_\mu$ are linearly independent. It’s not hard to prove that this is the case for all $n$.

**Definition 2.** The lexicographic order on partitions $\lambda, \mu \vdash n$ is defined as follows: $\lambda > \mu$ if for some $k > 0$

\[
\begin{align*}
\lambda_1 & = \mu_1, \\
\lambda_1 + \lambda_2 & = \mu_1 + \mu_2, \\
\vdots \\
\lambda_1 + \ldots + \lambda_{k-1} & = \mu_1 + \ldots + \mu_{k-1}, \\
\lambda_1 + \ldots + \lambda_k & > \mu_1 + \ldots + \mu_k.
\end{align*}
\]

Abbreviate $\chi_{\rho_\mu}$ by $\chi_\mu$ henceforth. Since the $\rho_\mu$ are permutation representations, we can calculate $\chi_\mu$ by counting fixed points. That is,

\[ \chi_\mu C_\lambda = \#\{\text{tabloids } T \mid \text{sh}(T) = \mu, \ w(T) = T\} \]

for any $w \in C_\lambda$.

**Proposition 1.** Let $\lambda, \mu \vdash n$. Then:

1. $\chi_\lambda(C_\lambda) \neq 0$.
2. $\chi_\mu(C_\lambda) \neq 0$ only if $\lambda \leq \mu$ in lexicographic order.

**Proof.** To show that $\chi_\lambda(C_\lambda) \neq 0$, let $w \in C_\lambda$; we must find a tabloid $T$ of shape $\lambda$ fixed by $w$. Indeed, we can take $T$ to be any tabloid whose blocks are the cycles of $w$. For example, if $w = (1 \ 3 \ 6)(2 \ 7)(4 \ 5) \in S_7$, then $T$ can be either of the following two tabloids:

\[
\begin{align*}
1 & \ 3 & \ 6 \\
2 & \ 7 \\
4 & \ 5
\end{align*}
\]

On the other hand, $w$ fixes a tabloid $T$ of shape $\mu$ if and only if every cycle of $w$ is contained in a row of $P$. In particular, the sum of any $r$ parts of $\lambda$ must be less than or equal to the sum of some $r$ parts of $\mu$, hence less than or equal to the sum of the first $r$ parts of $\mu$, which implies that $\lambda \leq \mu$. 

**Corollary 2.** The characters $\{\chi_\mu \mid \mu \vdash n\}$ form a basis for $Cl(G)$.

**Proof.** The number of these characters is dim $Cl(G)$. Moreover, Proposition $\Box$ implies that the $p(n) \times p(n)$ matrix $X = [\chi_\mu(C_\lambda)]_{\mu, \lambda \vdash n}$ is triangular, hence nonsingular.

We can transform the rows of the matrix $X$ into a list of irreducible characters of $S_n$ by applying the Gram-Schmidt process (measuring orthogonality, of course, with the inner product $\langle \cdot, \cdot \rangle_{S_n}$). Indeed, the triangularity of $X$ means that we will be able to label the irreducible characters of $S_n$ as

\[ \{\tilde{\chi}_\nu \mid \nu \vdash n\} \]
so that
\[ \langle \tilde{\chi}_\nu, \chi_\nu \rangle_G \neq 0, \]
\[ \langle \tilde{\chi}_\nu, \chi_\mu \rangle_G = 0 \quad \text{if } \nu < \mu. \]

**Example 2.** Recall the table of characters (1) of the representations \( \rho_\mu \) for \( n = 3 \). We will use this to produce the table of irreducible characters. For brevity, let’s omit the commas between the parts of partitions \( \mu \).

First, \( \chi(3) = [1,1,1] = \chi_{\text{triv}} \) is irreducible. We therefore call it \( \tilde{\chi}(3) \).

Second, for the character \( \chi(21) \), we observe that
\[ \langle \chi(21), \tilde{\chi}(3) \rangle_G = 1. \]
Applying Gram-Schmidt, we construct a character orthonormal to \( \tilde{\chi}(3) \):
\[ \tilde{\chi}(21) = \chi(21) - \tilde{\chi}(3) = [2,0,-1]. \]
Notice that this character is irreducible.

Finally, for the character \( \chi(111) \), we have
\[ \langle \chi(111), \tilde{\chi}(3) \rangle_G = 1, \]
\[ \langle \chi(111), \tilde{\chi}(21) \rangle_G = 2. \]
Accordingly, we apply Gram-Schmidt to obtain the character
\[ \tilde{\chi}(111) = \chi(111) - \tilde{\chi}(3) - 2\tilde{\chi}(21) = [1,-1,1] \]
which is 1-dimensional, hence irreducible. In summary, the complete list of irreducible characters, labeled so as to satisfy (2), is as follows:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( 111 )</th>
<th>( 21 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\chi}(3) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \tilde{\chi}(21) )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{\chi}(1,1,1) )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

\( \lambda \) \vdash 3 \quad \tilde{\chi}_\lambda = \chi_{\text{sign}} \]

To summarize our calculation, we have shown that
\[
\begin{bmatrix}
1 & 1 & 1 \\
3 & 1 & 0 \\
6 & 0 & 0 \\
\end{bmatrix} \mu \vdash 3 = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1 \\
\end{bmatrix} \chi_\mu = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & -1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
K_{\lambda,\mu} & \tilde{\chi}_\lambda \\
\end{bmatrix} \mu \vdash 3 \quad \chi_\mu = \sum_\lambda K_{\lambda,\mu} \tilde{\chi}_\lambda.
\]

The numbers \( K_{\lambda,\mu} \) are called the **Kostka numbers.** We will eventually find a combinatorial interpretation for them, which will imply easily that the matrix \( K \) is unitriangular.