Wednesday 4/16

Irreducible Characters

Theorem 1. *Let G be a finite group, and let (ρ, V) and (ρ', V') be finite-dimensional representations of G over $\mathbb{C}$.*

(i) *If ρ and ρ' are irreducible, then*

$$\langle \chi_\rho,\, \chi_{\rho'} \rangle_G \;=\; \begin{cases} 1 & \text{if } \rho \cong \rho' \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\rho_1, \dots, \rho_n$ are distinct irreducible representations and*

$$\rho \;=\; \bigoplus_{i=1}^{n} (\underbrace{\rho_i \oplus \cdot \oplus \rho_i}_{m_i}) \;=\; \bigoplus_{i=1}^{n} \rho_i^{\oplus m_i}$$

then

$$\langle \chi_\rho,\, \chi_{\rho_i} \rangle_G = m_i, \qquad \langle \chi_\rho,\, \chi_\rho \rangle_G = \sum_{i=1}^{n} m_i^2.$$

In particular, $\langle \chi_\rho,\, \chi_\rho \rangle_G = 1$ if and only if ρ is irreducible.

(iii) *If $\chi_\rho = \chi_{\rho'}$ then $\rho \cong \rho'$.*

(iv) *If $\rho_1, \dots, \rho_n$ is a complete list of irreducible representations of G, then*

$$\rho_{\text{reg}} \cong \bigoplus_{i=1}^{n} \rho_i^{\oplus \dim \rho_i}$$

and consequently

$$\sum_{i=1}^{n} (\dim \rho_i)^2 = |G|.$$

(v) *The irreducible characters (i.e., characters of irreducible representations) form an orthonormal basis for $C\ell(G)$. In particular, the number of irreducible characters equals the number of conjugacy classes of G.*

Proof. We proved everything last time except for the assertion that the irreducible characters span $C\ell(G)$. We will do this by showing that their orthogonal complement is zero.

Suppose that $f \in C\ell(G)$ is orthogonal to every ρ_i, i.e.,

$$\langle f,\, \chi_{\rho_i} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \chi_{\rho_i}(g) = 0.$$

We will show that $f = 0$.

For any representation ρ, define a map $T_\rho = T_{\rho,f} : V \to V$ by

$$T_\rho(v) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} g v.$$

I claim that T_ρ is G-equivariant. Indeed, for $h \in G$,

$$\begin{aligned} T_\rho(hv) &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}(gh)(v) \\ &= h\frac{1}{|G|}\sum_{g\in G}\overline{f(g)}(h^{-1}ghv) \\ &= h\frac{1}{|G|}\sum_{k=h^{-1}gh\in G}\overline{f(hkh^{-1})}(kv) \\ &= h\frac{1}{|G|}\sum_{k\in G}\overline{f(k)}(kv) && \text{(because } f \in C\ell(G)\text{)} \\ &= hT_\rho(v). \end{aligned}$$

Suppose now that ρ_i is irreducible. By Schur's Lemma, T_{ρ_i} is multiplication by a scalar. On the other hand, by the definition of f, we have

$$\begin{aligned} 0 \;=\; \langle f,\, \chi_{\rho_i}\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\chi_{\rho_i}(g) \\ &= \operatorname{tr}\left(\frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho_i(g)\right) \\ &= \operatorname{tr} T_{\rho_i}. \end{aligned}$$

Therefore $T_{\rho_i} = 0$ for every irreducible ρ_i. Also, T is additive on direct sums (that is, $T_{\rho\oplus\rho'} = T_\rho + T_{\rho'}$), so by Maschke's Theorem, $T_\rho = 0$ for every representation ρ. In particular

$$0 \;=\; T_{\rho_{\text{reg}}}(1_G) \;=\; \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}g.$$

It follows that $f(g) = 0$ for every $g \in G$, as desired. $\square$

One-Dimensional Characters

Let G be a group and ρ a one-dimensional representation; that is, ρ is a group homomorphism $G \to \mathbb{C}^\times$. Note that $\chi_\rho = \rho$. Also, if ρ' is another one-dimensional representation, then

$$\rho(g)\rho'(g) = (\rho \otimes \rho')(g)$$

for all $g \in G$. Thus the group $Ch(G) = \mathrm{Hom}(G, \mathbb{C}^\times)$ of all one-dimensional characters forms a group under pointwise multiplication. The trivial character is the identity of $Ch(G)$, and the inverse of a character ρ is its dual $\rho^* = \bar{\rho}$.

Definition 1. The **commutator** of two elements $a, b \in G$ is the element $[a, b] = aba^{-1}b^{-1}$. The subgroup of G generated by all commutators is called the **commutator subgroup**, denoted $[G, G]$.

It is simple to check that $[G, G]$ is in fact a normal subgroup of G. Moreover, $\rho([a, b]) = 1$ for all $\rho \in Ch(G)$ and $a, b \in G$. Therefore, the one-dimensional characters of G are precisely those of the quotient $G^{ab} = G/[G, G]$, the *abelianization* of G.

Accordingly, we would like to understand the characters of abelian groups.

Let G be an abelian group of finite order n. The conjugacy classes of G are all singleton sets (since $ghg^{-1} = h$ for all $g, h \in G$), so there are n distinct irreducible representations of G. On the other hand,

$$\sum_{\chi \text{ irreducible}} (\dim \chi)^2 = n$$

by Theorem 1 (iv), so in fact every irreducible character is 1-dimensional (and every representation of G is a direct sum of 1-dimensional representations).

Since a 1-dimensional representation equals its character, we just need to describe the homomorphisms $G \to \mathbb{C}^\times$.

The simplest case is that $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic. Write G multiplicatively, and let g be a generator. Then each $\chi \in Ch(G)$ is determined by its value on g, which must be some n^{th} root of unity. There are n possibilities for χ, so all the irreducible characters of G arise in this way, and in fact form a group isomorphic to G.

Now we consider the general case. Every abelian group G can be written as

$$G \cong \prod_{i=1}^{r} \mathbb{Z}/n_i\mathbb{Z}.$$

Let g_i be a generator of the i^{th} factor, and let ζ_i be a primitive $(n_i)^{th}$ root of unity. Then each character χ is determined by the numbers $j_1, \ldots, j_r$, where $j_i \in \mathbb{Z}/n_i\mathbb{Z}$ and $\chi(g_i) = \zeta_i^{j_i}$. for all i. By now, it should be evident that

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong G,$$

an isomorphism known as *Pontrjagin duality.* More generally, for any group G we have

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong G^{ab}. \tag{1}$$

This is quite useful when computing irreducible characters, because it tells us right away about the one-dimensional characters of an arbitrary group.

Example 1. Consider the case $G = \mathfrak{S}_n$. Certainly $[\mathfrak{S}_n, \mathfrak{S}_n] \subseteq \mathfrak{A}_n$, and in fact equality holds. (This is trivial for $n \leq 2$. If $n \leq 3$, then the equation $(a\ b)(b\ c)(a\ b)(b\ c) = (a\ b\ c)$ in $\mathfrak{S}_n$ (multiplying left to right) shows that $[\mathfrak{S}_n, \mathfrak{S}_n]$ contains every 3-cycle, and it is not hard to show that the 3-cycles generate the full alternating group.) Therefore (1) gives

$$\mathrm{Hom}(\mathfrak{S}_n, \mathbb{C}^\times) \cong \mathfrak{S}_n/\mathfrak{A}_n \cong \mathbb{Z}/2\mathbb{Z}$$

which says that χ_{triv} and χ_{sign} are the only one-dimensional characters of $\mathfrak{S}_n$.