Monday 4/7

Group Representations

Definition 1. Let G be a group and let $V \cong \mathbb{F}^n$ be a finite-dimensional vector space over a field $\mathbb{F}$. A **representation of** G **on** V is a group homomorphism $\rho : G \to GL(V)$. That is, for each $g \in G$ there is an invertible $n \times n$ matrix $\rho(g)$, satisfying

$$\rho(g)\rho(h) = \rho(gh) \qquad \forall g, h \in G.$$

(That's matrix multiplication on the left side of the equation, and group multiplication in G on the right.) The number n is called the **dimension** (or **degree**) of the representation.

- ρ specifies an action of G on V that respects its vector space structure.
- We often abuse terminology by saying that ρ is a representation, or that V is a representation, or that the pair (ρ, V) is a representation.
- ρ is a **permutation representation** if $\rho(g)$ is a permutation matrix for all $g \in G$.
- ρ is **faithful** if it is injective as a group homomorphism.

Example 1 (The regular representation)**.** Let G be a finite group with n elements, and let $\mathbb{F}G$ be the vector space of formal $\mathbb{F}$-linear combinations of elements of G: that is,

$$\mathbb{F}G = \left\{ \sum_{h \in G} a_h h \mid a_h \in \mathbb{F} \right\}.$$

Then there is a representation ρ_{reg} of G on $\mathbb{F}G$, called the **regular representation**, defined by

$$g\left(\sum_{h \in G} a_h h\right) = \sum_{h \in G} a_h (gh).$$

That is, g permutes the standard basis vectors of $\mathbb{F}G$ according to the group multiplication law.

Example 2 (The defining representation of $\mathfrak{S}_n$)**.** Let $G = \mathfrak{S}_n$, the symmetric group on n elements. Then we can represent each permutation $\sigma \in G$ by the permutation matrix with 1's in the positions $(i, \sigma(i))$ for every $i \in [n]$, and 0's elsewhere. For instance, the permutation $4716253 \in \mathfrak{S}_7$ corresponds to the permutation matrix

$$\begin{bmatrix} 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0 \end{bmatrix}.$$

Example 3. For any group G, the **trivial representation** is defined by $\rho_{\text{triv}}(g) = I_n$ (the $n \times n$ identity matrix).

Example 4. Let $G = \mathbb{Z}/k\mathbb{Z}$ be the cyclic group of order k, and let ζ be a k^{th} root of unity (not necessarily primitive). Then G has a 1-dimensional representation given by $\rho(x) = \zeta^x$.

Example 5. Let G act on a finite set X. Then there is an associated representation on $\mathbb{F}^X$, the vector space with basis X, given by

$$\rho(g)\left(\sum_{x \in X} a_x x\right) = \sum_{x \in X} a_x (g \cdot x).$$

For instance, the action of G on itself by left multiplication gives rise in this way to the regular representation.

Example 6. Let $G = D_n$, the dihedral group of order $2n$, i.e., the group of symmetries of a regular n-gon, given in terms of generators and relations by

$$\langle s, r \ : \ s^2 = r^n = 1, \ srs = r^{-1} \rangle.$$

There are a bunch of associated faithful representations of G.

First, we can regard s as a reflection and r as a rotation:

$$\rho(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \rho(r) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(where $\theta = 2\pi/n$). This is a faithful 2-dimensional representation.

Alternately, we can consider the actions of G on vertices, or on edges, or on opposite pairs, or on diameters. These are all faithful n-dimensional representations, except for the last — if n is even, then this representation is $2n$-dimensional and not faithful.

Example 7. The symmetric group $\mathfrak{S}_n$ has a nontrivial 1-dimensional representation, the **sign representation**, given by

$$\rho_{\text{sign}}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Note that $\rho_{\text{sign}}(g) = \det \rho_{\text{def}}(g)$, where ρ_{def} is the defining representation of $\mathfrak{S}_n$. In general, if ρ is any representation, then $\det \rho$ is a 1-dimensional representation. Note that

For you algebraists, a representation of G is the same thing as a left module over the group algebra $\mathbb{F}G$.

Example 8. Let (ρ, V) and (ρ', V') be representations of G, where $V \cong \mathbb{F}^n$, $V' \cong \mathbb{F}^m$. The **direct sum** $\rho \oplus \rho' : G \to GL(V \oplus V')$ is defined by

$$(\rho \oplus \rho')(g)(v + v') = \rho(g)(v) + \rho'(g)(v')$$

for $v \in V$, $v' \in V'$. In terms of matrices, $(\rho \oplus \rho')(g)$ is a block-diagonal matrix

$$\left[\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \rho'(g) \end{array}\right].$$

Isomorphisms and Homomorphisms

When two representations are the same? More generally, what is a map between representations?

Definition 2. Let (ρ, V) and (ρ', V') be representations of G. A linear transformation $\phi : V \to V'$ is **G-equivariant** if $g\phi = \phi g$.

Equivalently, $g \cdot \phi(v) = \phi(g \cdot v)$ for all $g \in G$, $v \in V$. [Or, more precisely if less concisely: $\rho'(g) \cdot \phi(v) = \phi(\rho(g) \cdot v)$.]

If you insist, this is equivalent to the condition that the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\phi"'] & V' \end{tikzcd}$$

Abusing notation as usual, we might write $\phi : \rho \to \rho'$.

In the language of modules, these are just G-module homomorphisms. Accordingly, the vector space of all G-equivariant maps $V \to V'$ is denoted $\operatorname{Hom}_G(V, V')$. This is itself a representation of G.

Example 9. One way in which G-equivariant transformations occur is when an action "naturally" induces another action. For instance, consider the permutation action of $\mathfrak{S}_4$ on the vertices of K_4, which induces a 4-dimensional representation ρ_v. However, this action naturally determines an action on the six edges of K_4, which in turn induces a 6-dimensional representation ρ_e. This is to say that there is a G-equivariant transformation $\rho_v \to \rho_e$.

Definition 3. Two representations (ρ, V) and (ρ', V') of G are **isomorphic** if there is a G-equivariant map $\phi : V \to V'$ that is a vector space isomorphism.

Example 10. Let $\mathbb{F}$ be a field of characteristic $\neq 2$, and let $V = \mathbb{F}^2$, with standard basis $\{e_1, e_2\}$. Let $G = \mathfrak{S}_2 = \{12, 21\}$. The defining representation $\rho = \rho_{\text{def}}$ of G on V is given by

$$\rho(12) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(21) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

On the other hand, the representation $\sigma = \rho_{\text{triv}} \oplus \rho_{\text{sign}}$ is given on V by

$$\sigma(12) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma(21) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These two representations are in fact isomorphic. Indeed, ρ acts trivially on $\operatorname{span}\{e_1 + e_2\}$ and acts by -1 on $\operatorname{span}\{e_1 - e_2\}$. Since these two vectors form a basis of V, one can check that

$$\phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

is an isomorphism $\rho \to \sigma$.

Our goal is to classify representations up to isomorphism. As we will see, we can do this without having to worry about every coordinate of every matrix $\rho(g)$ — all we really need to know is the *trace* of $\rho(g)$, known as the **character** of a representation. For instance, in this last example, we can detect the isomorphism $\rho \cong \sigma$ by observing that

$$\operatorname{tr}(\rho(12)) = \operatorname{tr}(\sigma(12)) = 2, \qquad \operatorname{tr}(\rho(21)) = \operatorname{tr}(\sigma(21)) = 0.$$