Friday 3/7

Counting Regions of Hyperplane Arrangements

For $\mathcal{A} \subset \mathbb{R}^n$ a real hyperplane arrangement, we defined last time

$$\begin{aligned} r(\mathcal{A}) &= \text{ \# of regions of } \mathcal{A}, \\ b(\mathcal{A}) &= \text{ \# of relatively bounded regions of } \mathcal{A}. \end{aligned}$$

Also, we proved the following recurrence.

Proposition 1. *For $H \in \mathcal{A}$, let*

$$\begin{aligned} \mathcal{A}' &= \mathcal{A} \setminus \{H\}, \\ \mathcal{A}'' &= \mathcal{A}^H = \{W \cap H \mid W \in \mathcal{A},\ W \not\supseteq H\}. \end{aligned}$$

Then

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'') \tag{1}$$

and

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \text{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}', \\ 0 & \text{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}' + 1. \end{cases} \tag{2}$$

Proposition 2 (Deletion/Restriction)**.** *Let $\mathcal{A}$ be a real arrangement and $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$. Then*

$$\chi_{\mathcal{A}}(k) = \chi_{\mathcal{A}'}(k) - \chi_{\mathcal{A}''}(k). \tag{3}$$

Proof. First, we establish Whitney's formula for the characteristic polynomial. Consider the interval $[\hat{0}, x]$. The atoms in this interval are the hyperplanes of $\mathcal{A}$ containing x, and they form a lower crosscut of $[\hat{0}, x]$. Therefore, the crosscut theorem (3/3/08) says that

$$\mu(\hat{0}, x) = \sum_{Y \subset \mathcal{A}:\ x = \bigcap Y} (-1)^{|Y|}. \tag{4}$$

Plugging (4) into the definition of the characteristic polynomial, we get

$$\begin{aligned} \chi_{\mathcal{A}}(k) &= \sum_{x \in L(\mathcal{A})} \sum_{Y \subset \mathcal{A}:\ x = \bigcap Y} (-1)^{|Y|} k^{\dim x} \\ &= \sum_{Y \subset \mathcal{A}:\ \bigcap Y \neq 0} (-1)^{|Y|} k^{\dim \bigcap Y} \\ &= \sum_{\text{central } \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} k^{\dim \mathcal{A} - \operatorname{rank} \mathcal{B}} \qquad (5) \end{aligned}$$

which is Whitney's formula.

Now, split the sum in (5) into two pieces, depending on whether or not $H \in \mathcal{B}$. First,

$$\sum_{\text{central } \mathcal{B} \subseteq \mathcal{A}:\ H \notin \mathcal{B}} (-1)^{|\mathcal{B}|} k^{\dim \mathcal{A} - \operatorname{rank} \mathcal{B}} = \sum_{\text{central } \mathcal{B} \subseteq \mathcal{A}'} (-1)^{|\mathcal{B}|} k^{\dim \mathcal{A} - \operatorname{rank} \mathcal{B}} = \chi_{\mathcal{A}'}(k). \tag{6}$$

Second, suppose $\mathcal{B} \subseteq \mathcal{A}$ is a central arrangement containing H. This is a little trickier because hyperplanes that are distinct in $\mathcal{A}$ do not necessarily correspond to distinct hyperplanes in $\mathcal{A}''$, so we have to do a bit more work to rewrite the other subsum of (5) as a sum over central subarrangements of $\mathcal{A}''$. (Stanley's notes do not discuss this issue.) Define a map $\pi : \mathcal{A}' \to \mathcal{A}''$ by $\pi(x) = x \cap H$; then

$$\sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \mathcal{B}\text{ central, } H\in\mathcal{B}}} (-1)^{|\mathcal{B}|} k^{\dim\mathcal{A}-\operatorname{rank}\mathcal{B}}$$

$$= \sum_{\mathcal{C}\subseteq\mathcal{A}''\text{central}} \quad \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}''\\ H\in\mathcal{B},\ \pi(\mathcal{B})=\mathcal{C}}} (-1)^{|\mathcal{B}|} k^{\dim\mathcal{A}''-\operatorname{rank}\mathcal{C}}$$

$$= -\sum_{\substack{\mathcal{C}\subseteq\mathcal{A}''\text{central}\\ \mathcal{C}=\{H_1'',\dots,H_s''\}}} k^{\dim\mathcal{A}''-\operatorname{rank}\mathcal{C}} \left(\sum_{\emptyset\neq\mathcal{B}_1\subseteq\pi^{-1}H_1''} \cdots \sum_{\emptyset\neq\mathcal{B}_1\subseteq\pi^{-1}H_s''} (-1)^{|\mathcal{B}_1|}\cdots(-1)^{|\mathcal{B}_s|} \right)$$

$$= -\sum_{\substack{\mathcal{C}\subseteq\mathcal{A}''\text{central}\\ |\mathcal{C}|=s}} k^{\dim\mathcal{A}''-\operatorname{rank}\mathcal{C}}(-1)^s \quad = \quad -\chi_{\mathcal{A}''}(k). \tag{7}$$

Now the desired recurrence follows from (5), (6) and (7). □

Theorem 3 (Zaslavsky 1975)**.** *Let $\mathcal{A}$ be a real hyperplane arrangement. Then*

$$r(\mathcal{A}) = (-1)^{\dim\mathcal{A}}\chi_{\mathcal{A}}(-1), \tag{8}$$

$$c(\mathcal{A}) = (-1)^{\operatorname{rank}\mathcal{A}}\chi_{\mathcal{A}}(1). \tag{9}$$

Sketch of proof. Compare the recurrences for r and c proved last time with those for these evaluations of the characteristic polynomial (from Proposition 2). □

Corollary 4. *Let $\mathcal{A}\subset\mathbb{R}^n$ be a central, essential hyperplane arrangement, so that $L(\mathcal{A})$ is a geometric lattice. Let M be the corresponding matroid. Then*

$$r(\mathcal{A}) = T(M;2,0), \qquad c(\mathcal{A}) = T(M;0,0) = 0.$$

Proof. Combine Zaslavsky's theorem with the formula $\chi_{\mathcal{A}}(k) = (-1)^n T(M;1-k,0)$. □

Example 1. Let $m \geq n$, and let $\mathcal{A}$ be an arrangement of m linear hyperplanes in general position in $\mathbb{R}^n$. The corresponding matroid M is $U_n(m)$, whose rank function is

$$r(A) = \min(n, |A|)$$

for $A \subseteq [m]$. Therefore

$$\begin{aligned}
r(\mathcal{A}) \;=\; T(M;2,0) &= \sum_{A\subseteq[m]} (1-1)^{n-r(A)}(0-1)^{|A|-r(A)} \\
&= \sum_{A\subseteq[m]} (-1)^{|A|-r(A)} \\
&= \sum_{k=0}^{m} \binom{m}{k}(-1)^{k-\min(n,k)} \\
&= \sum_{k=0}^{n} \binom{m}{k} \;+\; \sum_{k=n+1}^{m} \binom{m}{k}(-1)^{k-n} \\
&= \sum_{k=0}^{n} \binom{m}{k}(1-(-1)^{k-n}) \;+\; \sum_{k=0}^{m} \binom{m}{k}(-1)^{k-n} \\
&= \sum_{k=0}^{n} \binom{m}{k}(1-(-1)^{k-n}) \\
&= 2\left(\binom{m}{n-1} + \binom{m}{n-3} + \cdots\right).
\end{aligned}$$

For instance, if $n = 3$ then

$$r(\mathcal{A}) = 2\left(\binom{m}{2} + \binom{m}{0}\right) = m^2 - m + 2.$$

Notice that this is *not* the same as the formula we obtained last time for the number of regions formed by m affine lines in general position in $\mathbb{R}^2$.

Another Interpretation of the Characteristic Polynomial

Let $\mathbb{F}_q$ be the finite field of order q, and let $\mathcal{A} \subset \mathbb{F}_q^n$ be a hyperplane arrangement. The "regions" of $\mathbb{F}_q^n \setminus \mathcal{A}$ are just its points (assuming, if you wish, that we endow K^n with the discrete topology). The following result is implicit in the work of Crapo and Rota (1970) and was stated explicitly by Athanasiadis (1996):

Proposition 5. $|\mathbb{F}_q^n \setminus \mathcal{A}| = \chi_{\mathcal{A}}(q)$.

Proof. By inclusion-exclusion, we have

$$|\mathbb{F}_q^n \setminus \mathcal{A}| \;=\; \sum_{\mathcal{B}\subseteq\mathcal{A}} (-1)^{|\mathcal{B}|} \left|\bigcap \mathcal{B}\right|.$$

If $\mathcal{B}$ is not central, then by definition $|\bigcap \mathcal{B}| = 0$. Otherwise, $|\bigcap \mathcal{B}| = q^{n-\operatorname{rank}\mathcal{B}}$. So the sum becomes Whitney's formula for $\chi_{\mathcal{A}}(q)$. □

This fact has a much more general application, which was systematically mined by Athanasiadis (1996). Let $\mathcal{A} \subset \mathbb{R}^n$ be an arrangement defined over the integers (i.e., such that the normal vectors to its hyperplanes lie in $\mathbb{Z}^n$). For a prime p, let $\mathcal{A}_p \subset \mathbb{F}_p^n$ be the arrangement defined by regarding the coordinates of the normal vectors as numbers modulo p. If p is sufficiently large, then it will be the case that $L(\mathcal{A}_p) \cong L(\mathcal{A})$. In this case we say that $\mathcal{A}$ **reduces correctly modulo** p. But that means that we can compute the characteristic polynomial of $\mathcal{A}$ by counting the points of $\mathcal{A}_p$ as a function of p, for large enough p.