Monday 3/3

Crosscuts

Recall that the **Möbius algebra** $A(L)$ of a lattice L is the vector space of formal $\mathbb{C}$-linear combinations of elements of L, with multiplication given by the meet operation.

We showed last time that $A(L) \cong \mathbb{C}^{|L|}$ as rings, because the elements $\varepsilon_x = \sum_{y \leq x} \mu(y,x) y$ satisfy $\varepsilon_x \varepsilon_y = \delta_{xy} \varepsilon_x$ (where δ_{xy} is the Kronecker delta).

Definition 1. Let L be a lattice.
— An **upper crosscut** of L is a set $X \subset L \setminus \{\hat{1}\}$ such that if $y \in L \setminus X \setminus \{\hat{1}\}$, then $y < x$ for some $x \in X$.
— A **lower crosscut** of L is a set $X \subset L \setminus \{\hat{0}\}$ such that if $y \in L \setminus X \setminus \{\hat{0}\}$, then $y > x$ for some $x \in X$.

Proposition 1 (Rota's Crosscut Theorem)**.** *Let L be a finite lattice and let X be an upper crosscut. Then*

$$\mu(L) = \sum_{Y \subseteq X:\ \bigwedge Y = \hat{0}} (-1)^{|Y|}. \tag{1a}$$

Dually, if X is a lower crosscut, then

$$\mu(L) = \sum_{Y \subseteq X:\ \bigvee Y = \hat{1}} (-1)^{|Y|}. \tag{1b}$$

Proof. Let $x \in L$. We have the following equation in the Möbius algebra of L:

$$\hat{1} - x \;=\; \left(\sum_{y \in L} \varepsilon_y\right) - \left(\sum_{y \leq x} \varepsilon_y\right) \;=\; \left(\sum_{y \not\leq x} \varepsilon_y\right).$$

Therefore

$$\prod_{x \in X} (\hat{1} - x) \;=\; \prod_{x \in X} \left(\sum_{y \not\leq x} \varepsilon_y\right) \;=\; \sum_{y \in Y} \varepsilon_y$$

where $Y = \{y \in L \mid y \not\leq x \text{ for all } x \in X\}$. But $Y = \{\hat{1}\}$ because X is an upper crosscut. That is,

$$\prod_{x \in X} (\hat{1} - x) \;=\; \varepsilon_{\hat{1}} \;=\; \sum_{y \in L} \mu(y, \hat{1}) y. \tag{2}$$

On the other hand

$$\prod_{x \in X} (\hat{1} - x) \;=\; \sum_{A \subseteq X} (-1)^{|A|} \bigwedge A. \tag{3}$$

Now extracting the coefficient of $\hat{0}$ on the right-hand sides of (2) and (3) yields (1a). The proof of (1b) is similar. $\square$

We already know that $\mu(\mathscr{B}_n) = (-1)^n$. What if L is distributive but not Boolean?

Proposition 2. *Let L be a distributive lattice that is not Boolean. Then $\mu(L) = 0$.*

Proof. The set X of atoms of L is a lower crosscut. Since L is not Boolean, it has a join-irreducible element $b \notin X$. But $b \not\leq \bigvee X$ because $b \not\leq a$ for all $a \in X$ (this is Lemma 2 of the lecture notes from 1/28/07). In particular, $\bigvee X \neq \hat{1}$, so the sum on the right-hand side of (1b) is empty. $\square$

More generally, $\mu(L) = 0$ if L is a lattice in which $\hat{1}$ is not a join of atoms (or, dually, if $\hat{0}$ is not a meet of coatoms).

The crosscut theorem will also be useful in studying hyperplane arrangements.

A Topological Application

Proposition 3. *Let P be a chain-finite, bounded poset, and let*

$$c_i = \left|\{\hat{0} = x_0 < x_1 < \cdots < x_i = \hat{1}\}\right|,$$

the number of chains of length i between $\hat{0}$ and $\hat{1}$. Then

$$\mu(P) = \sum_i (-1)^i c_i. \tag{4}$$

Proof. The incidence algebra makes the proof almost trivial. Recall that $c_i = \eta^i(\hat{0},\hat{1}) = (\zeta-1)^i(\hat{0},\hat{1})$. Since sufficiently high powers of η vanish,

$$(1-\eta)\left(\sum_{i=0}^{\infty}\eta^i\right) = 1$$

is a perfectly good equation in $I(P)$. Therefore

$$\sum_{i=0}^{\infty} c_i \;=\; \sum_{i=0}^{\infty}\eta^i(\hat{0},\hat{1}) \;=\; (1-\eta)^{-1}(\hat{0},\hat{1}) \;=\; \zeta^{-1}(\hat{0},\hat{1}) \;=\; \mu(\hat{0},\hat{1}). \qquad \square$$

The expression (4) looks like an Euler characteristic. Indeed, let P be a finite poset. The **order complex** of P is defined as the simplicial complex $\Delta = \Delta(P)$ whose vertices are elements of P and whose faces are chains of P. Then Proposition 3 implies that $\mu(P)$ is the reduced Euler characteristic of Δ.

Another nice fact is the following result due to J. Folkman (1966), whose proof used the crosscut theorem.

Theorem 4. *Let L be a geometric lattice of rank r, and let $P = L \setminus \{\hat{0},\hat{1}\}$. Then*

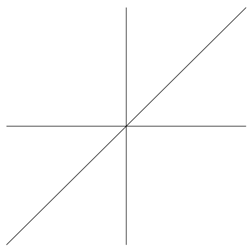
$$\tilde{H}_i(\Delta(P),\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{|\mu(L)|} & \text{if } i = r-2, \\ 0 & \text{otherwise} \end{cases}$$

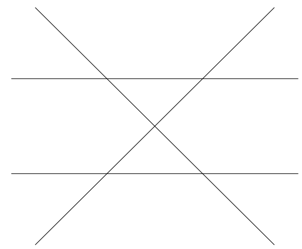
where $\tilde{H}_i$ denotes reduced simplicial homology.

Hyperplane Arrangements

Definition 2. Let K be a field and $n \geq 1$. A **linear hyperplane** in K^n is a vector subspace of codimension 1. An **affine hyperplane** is a translate of a linear hyperplane. A **hyperplane arrangement** $\mathcal{A}$ is a finite collection of (distinct) hyperplanes. The number n is called the **dimension** of $\mathcal{A}$.

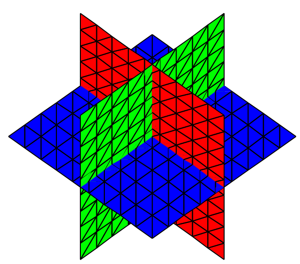
Example 1. The left-hand arrangement $\mathcal{A}_1$ is linear; it consists of the lines $x = 0$, $y = 0$, and $x = y$. The right-hand arrangement $\mathcal{A}_2$ is affine; it consists of the four lines $x = y$, $x = -y$, $y = 1$ and $y = -1$.





Each hyperplane is the zero set of some linear form, so their union is the zero set of the product of those s linear forms. We can specify an arrangement concisely by that product, called the **defining polynomial** of $\mathcal{A}$ (as an algebraic variety, in fact). For example, the defining polynomials of $\mathcal{A}_1$ and $\mathcal{A}_2$ are $xy(x - y)$ and $xy(x - y - 1)$ respectively.

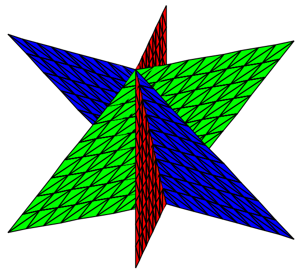
Example 2. Here are some 3-D arrangements (pictures produced using Maple). The *Boolean arrangement* $\mathscr{B}_n$ consists of the coordinate hyperplanes in n-space, so its defining polynomial is $x_1x_2 \ldots x_n$. Here's $\mathscr{B}_3$.



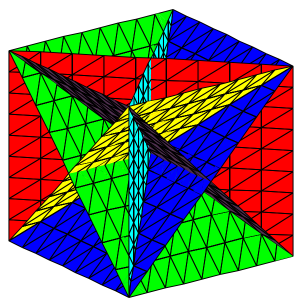
The *braid arrangement* $\mathscr{B}_n$ consists of the $\binom{n}{2}$ hyperplanes $x_i - x_j$ in n-space, so its defining polynomial is

$$\prod_{1 \leq i < j \leq n} x_i - x_j.$$

Here's Br_3.



Every hyperplane in $\mathscr{B}_n$ contains the line spanned by the all-ones vector. If we project $\mathbb{R}^4$ to the quotient by that line, then $\mathcal{A}_4$ ends up looking like this:



The Intersection Poset

Definition 3. Let $\mathcal{A} \subset K^n$ be an arrangement. Its **intersection poset** $L(\mathcal{A})$ is the poset of all intersections of subsets of $\mathcal{A}$, ordered by reverse inclusion. This poset always has a $\hat{0}$ element, namely K^n. It has a $\hat{1}$ element if and only if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$; such an arrangement is called **central**.

Proposition 5. *Let $\mathcal{A} \subset K^n$ be an arrangement. The following are equivalent:*

- *$\mathcal{A}$ is central.*
- *$\mathcal{A}$ is a translation of a linear arrangement.*
- *$L(\mathcal{A})$ is a geometric lattice.*

Proof. Linear arrangements are central because every hyperplane contains $\vec{0} \in K^n$. Conversely, if $\mathcal{A}$ is central and $p \in \bigcap_{H \in \mathcal{A}} H$, then translating everything by $-p$ produces a linear arrangement.

If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is bounded. It is a join-semilattice, with join given by intersection, hence it is a lattice. Indeed, it is a geometric lattice (it is clearly atomic, and it is submodular because it is a sublattice of the chain-finite modular lattice $L(K^n)^*$ — that is, the dual of the lattice of all subspaces of K^n). $\square$

When $\mathcal{A}$ is central (we may as well assume linear), the matroid associated with $L(\mathcal{A})$ is naturally represented by the normal vectors to the hyperplanes in $\mathcal{A}$.

Therefore, all of the tools we have developed for looking at lattices and matroids can be applied to study hyperplane arrangements.

The dimension of an arrangement cannot be inferred from the intersection poset. For example, if $\mathcal{A}_1$ is as above, then $L(\mathcal{A}_1) \cong L(Br_3)$ but $\dim \mathcal{A}_1 = 2$ and $\dim Br_3 = 3$. A more useful invariant of $\mathcal{A}$ is its **rank** $\operatorname{rank} \mathcal{A}$, defined as the rank of $L(\mathcal{A})$. Equivalently, define $W \subset K^n$ to be the subspace spanned by the normal vectors v_i. Then $\operatorname{rank} \mathcal{A} = \dim W$.

Definition 4. An arrangement $\mathcal{A}$ is **essential** if $\operatorname{rank} \mathcal{A} = \dim \mathcal{A}$. In general, the **essentialization** $\operatorname{ess}(\mathcal{A})$ is the arrangement

$$\{H \cap W \mid H \in \mathcal{A}\} \; \subset \; W.$$

Equivalently, if $V = W^\perp = \bigcap_{H \in \mathcal{A}} H$, then $\operatorname{rank} \mathcal{A} = n - \dim V$, and we could define the essentialization of $\mathcal{A}$ as a quotient:

$$\{H/V \mid H \in \mathcal{A}\} \; \subset \; K^n/V\}.$$

Observe that $\operatorname{ess}(\mathcal{A})$ is essential, and that $L(\mathcal{A})$ is naturally isomorphic to $L(\operatorname{ess}(\mathcal{A}))$.