Friday 2/29

More on the Characteristic Polynomial

Definition 1. Let P be a finite graded poset with rank function r, and suppose that $r(\hat{1}) = n$. The *characteristic polynomial* of P is defined as

$$\chi(P;x) \; = \; \sum_{z \in P} \mu(\hat{0}, z) x^{n-r(z)}.$$

Theorem 1. *Let L be a geometric lattice with atoms E. Let M be the corresponding matroid on E, and r its rank function. Then*

$$\chi(L;x) \; = \; (-1)^{r(M)} T(M;\, 1-x, 0).$$

(This was proved last time.)

Example 1. Let G be a simple graph with n vertices and c components so that its graphic matroid $M(G)$ has rank $n-c$. Let L be the geometric lattice corresponding to M. The flats of L are the *(vertex-)induced subgraphs* of G: the subgraphs H such that if $e = xy \in E(G)$, and x, y are in the same component of H, then $e \in E(H)$. We have seen before that the chromatic polynomial of G is

$$\chi(G;k) = (-1)^{n-c}\, k^c\, T(G,\, 1-k, 0).$$

Combining this with Theorem 1, we see that

$$\chi(G;k) = k^c\, \chi(L;k)$$

so there is not too much inconsistency between these two uses of the symbol χ.

The characteristic polynomial is particularly important in studying hyperplane arrangements (coming soon).

Möbius Functions of Lattices

Theorem 2. *The Möbius function of a geometric lattice alternates in sign.*

Proof. Let L be a geometric lattice with atoms E. Let M be the corresponding matroid on E, and r its rank function. Substituting $x = 0$ in the definition of the characteristic polynomial and in the formula of Theorem 1 gives

$$\mu(L) \; = \; \chi(L;\, 0) \; = \; (-1)^{r(M)} T(M;\, 1, 0).$$

But $T(M;\, 1, 0) \geq 0$ for every matroid M, because $T(M;\, x, y) \in \mathbb{N}[x, y]$. Meanwhile, every interval $[\hat{0}, z] \subset L$ is a geometric lattice, so the sign of $\mu(\hat{0}, z)$ is the same as that of $(-1)^{r(z)}$ (or zero). □

In fact, more is true: the Möbius function of any *semimodular* (not necessarily atomic) lattice alternates in sign. This can be proven algebraically using tools we're about to develop (Stanley, Prop. 3.10.1) or combinatorially, by intepreting $(-1)^{r(M)}\mu(L)$ as enumerating *R-labellings* of L; see Stanley, §§3.12–3.13.

It is easier to compute the Möbius function of a lattice than of an arbitrary poset. The main technical tool is the following ring.

Definition 2. Let L be a lattice. The **Möbius algebra** $A(L)$ is the vector space of formal $\mathbb{C}$-linear combinations of elements of L, with multiplication given by the meet operation. (So $\hat{1}$ is the multiplicative unit of $A(L)$.)

For example, if $L = \mathscr{B}_n$ then $A(L) \cong \mathbb{C}[x_1, \dots, x_n]/(x_1^2 - x_1, \dots, x_n^2 - x_n)$. In general, the elements of L form a vector space basis of $A(L)$ consisting of *idempotents* (since $x \wedge x = x$ for all $x \in L$).

It looks like $A(L)$ could have a complicated structure, but actually...

Proposition 3. *$A(L) \cong \mathbb{C}^{|L|}$ as rings.*

Proof. This is just an application of Möbius inversion. For $x \in L$, define

$$\varepsilon_x = \sum_{y \leq x} \mu(y,x) y.$$

By Möbius inversion

(1) $$x = \sum_{y \leq x} \varepsilon_y.$$

For $x \in L$, let $\mathbb{C}_x$ be a copy of $\mathbb{C}$ with unit 1_x, so we can identify $\mathbb{C}^{|L|}$ with $\prod_{x \in L} \mathbb{C}_x$.

Define a $\mathbb{C}$-linear map $\phi : A(L) \to \mathbb{C}^{|L|}$ by $\varepsilon_x \mapsto 1_x$. This is a vector space isomorphism, and by (1) we have

$$\phi(x)\phi(y) \;=\; \phi\left(\sum_{w \leq x} \varepsilon_w\right) \phi\left(\sum_{z \leq y} \varepsilon_z\right) \;=\; \left(\sum_{w \leq x} 1_w\right)\left(\sum_{z \leq y} 1_z\right) \;=\; \sum_{v \leq x \wedge y} 1_v \;=\; \phi(x \wedge y)$$

so in fact ϕ is a ring isomorphism. ☐

The reason the Möbius algebra is useful is that it lets us compute $\mu(x,y)$ more easily by summing over a cleverly chosen *subset* of $[x,y]$, rather than the entire interval.

Proposition 4. *Let L be a finite lattice with at least two elements. Then for every $a \in L \setminus \{\hat{1}\}$ we have*

$$\sum_{x : x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0.$$

Proof. On the one hand

$$a\varepsilon_1 \;=\; \left(\sum_{b \leq a} \varepsilon_b\right) \varepsilon_{\hat{1}} \;=\; 0.$$

On the other hand

$$a\varepsilon_1 \;=\; a\left(\sum_{x \in L} \mu(x,\hat{1}) x\right) \;=\; \sum_{x \in L} \mu(x,\hat{1}) x \wedge a.$$

Now take the coefficient of $\hat{0}$. ☐

A corollary of Proposition 4 is the useful formula

(2) $$\mu(L) \;=\; \mu_L(\hat{0},\hat{1}) \;=\; -\sum_{\substack{x \neq \hat{0}: \\ x \wedge a = \hat{0}}} \mu(x,\hat{1})$$

Example 2. Let $a = \{[n-1],\ \{n\}\} \in \Pi_n$. Then the partitions x that show up in the sum of (2) are the atoms whose non-singleton block is $\{i,n\}$ for some $i \in [n-1]$. For each such x, the interval $[x,\hat{1}] \subset \Pi_n$ is isomorphic to Π_{n-1}, so (2) gives

$$\mu(\Pi_n) \;=\; -(n-1)\mu(\Pi_{n-1})$$

from which it follows by induction that

$$\mu(\Pi_n) = (-1)^{n-1}(n-1)!.$$

(Wasn't that easy?)

Example 3. Let $L = L_n(q)$, and let $A = \{(v_1, \dots, v_n) \in \mathbb{F}_q^n \mid v_n = 0\}$. This is a codimension-1 subspace in $\mathbb{F}_q^n$, hence a coatom in L. If X is a nonzero subspace such that $X \cap A = 0$, then X must be a line spanned by some vector $(x_1, \dots, x_n)$ with $x_n \neq 0$. We may as well assume $x_n = 1$ and choose $x_1, \dots, x_{n-1}$ arbitrarily, so there are q^{n-1} such lines. Moreover, the interval $[X, \hat{1}] \subset L$ is isomorphic to $L_{n-1}(q)$. Therefore

$$\mu(L_n(q)) \;=\; -q^{n-1}\mu(L_{n-1}(q))$$

and by induction

$$\mu(L_n(q)) \;=\; (-1)^n q^{\binom{n}{2}}.$$