Friday 2/22

The Chromatic Polynomial

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a function $f : V \to [k]$; the coloring is *proper* if $f(v) \neq f(w)$ whenever $vw \in E$. The *chromatic function* of G is defined as

$$\chi(G;\ k) = \#\text{ of proper colorings of } G\}.$$

Theorem 1. *Let G be a graph with n vertices and c components. Let*

$$\tilde{\chi}(G;\ k) \ = \ (-1)^{n-c} k^c T(G,\ 1-k, 0).$$

Then $\tilde{\chi}(G;\ k) = \chi(G;\ k)$.

Proof. First, we show that the chromatic function satisfies the recurrence

(1) $\chi(G;\ k) = k^n$ if $E = \emptyset$;

(2) $\chi(G;\ k) = 0$ if G has a loop;

(3) $\chi(G;\ k) = (k-1)\chi(G/e;\ k)$ if e is a coloop;

(4) $\chi(G;\ k) = \chi(G-e;\ k) - \chi(G/e;\ k)$ otherwise.

If $E = \emptyset$ then every one of the k^n colorings of G is proper, and if G has a loop then it has no proper colorings, so (1) and (2) are easy.

Suppose $e = xy$ is not a loop. Let f be a proper k-coloring of $G - e$. If $f(x) = f(y)$, then we can identify x and y to obtain a proper k-coloring of G/e. If $f(x) \neq f(y)$, then f is a proper k-coloring of G. So (4) follows.

This argument applies even if e is a coloop. In that case, however, the component H of G containing e becomes two components H' and H'' of $G - e$, whose colorings can be chosen independently of each other. So the probability that $f(x) = f(y)$ in any proper coloring is $1/k$, implying (3).

(A corollary, by induction on $|V|$, is that $\chi(G;\ k)$ is a polynomial in k, and thus has the right to be called the *chromatic polynomial* of G.)

The graph $G - e$ has n vertices and either $c + 1$ or c components, according as e is or is not a coloop. Meanwhile, G/e has $n - 1$ vertices and c components. By the recursive definition of the Tutte polynomial

$$\begin{aligned}
\tilde{\chi}(G;\ k) \ &= \ (-1)^{n-c} k^c T(G,\ 1-k, 0) \\
&= \begin{cases} k^n & \text{if } E = \emptyset, \\ 0 & \text{if } e \text{ is a loop,} \\ (1-k)(-1)^{n+1-c} k^c T(G/e,\ 1-k, 0) & \text{if } e \text{ is a coloop,} \\ (-1)^{n-c} k^c \left(T(G-e,\ 1-k, 0) + T(G/e,\ 1-k, 0)\right) & \text{otherwise} \end{cases} \\
&= \begin{cases} k^n & \text{if } E = \emptyset, \\ 0 & \text{if } e \text{ is a loop,} \\ (k-1)\chi(G/e;\ k) & \text{if } e \text{ is a coloop,} \\ \chi(G-e;\ k) - \chi(G/e;\ k) & \text{otherwise} \end{cases}
\end{aligned}$$

which is exactly the recurrence satisfied by the chromatic polynomial. This proves the theorem. ☐

This result raises the question of what this specialization of $T(M)$ means in the case that M is a an arbitrary (not necessarily graphic) matroid. Stay tuned!

Acyclic Orientations

An *orientation* D of a graph $G = (V, E)$ is an assignment of a direction to each edge $xy \in E$ (either $\vec{xy}$ or $\vec{yx}$). A *directed cycle* is a sequence $(x_0, x_1, \ldots, x_{n-1})$ of vertices such that $\vec{x_i x_{i+1}}$ is a directed edge for every i. (Here the indices are taken modulo n.)

An orientation is *acyclic* if it has no directed cycles. Let $A(G)$ be the set of acyclic orientations of G, and let $a(G) = |A(G)|$.

Theorem 2 (Stanley 1973)**.** *For every graph G on n vertices, we have*

$$a(G) = T(G;\, 2, 0) = (-1)^{n-1}\chi(G;\, -1).$$

Proof. The second equality is a consequence of Theorem 1. Plugging $x = 2$ and $y = 0$ into the Definition of the Tutte polynomial, we obtain the recurrence we need to establish in order to prove the first equality:

(A1) If $E = \emptyset$, then $a(G) = 1$.
(A2a) If $e \in E$ is a loop, then $a(G) = 0$.
(A2b) If $e \in E$ is a coloop, then $a(G) = 2a(G/e)$.
(A3) If $e \in E$ is neither a loop nor a coloop, then $a(G) = a(G - e) + a(G/e)$.

(A1) holds because the number of orientations of G is $2^{|V|}$, and any orientation of a forest (in particular, an edgelesss graph) is acyclic.

For **(A2a)**, note that if G has a loop then it cannot possibly have an acyclic orientation.

If G has a coloop e, then e doesn't belong to any cycle of G, so any acyclic orientation of G/e can be extended to an acyclic orientation of G by orienting e in either direction, proving **(A2b)**.

The trickiest part is **(A3)**. Fix an edge $e = xy \in E(G)$. For each orientation D of G, let $\tilde{D}$ be the orientation produced by reversing the direction of e, and let

$$A_1 = \{D \in A(G) \mid \tilde{D} \in A(G)\},$$
$$A_2 = \{D \in A(G) \mid \tilde{D} \notin A(G)\}.$$

Clearly $a(G) = |A_1| + |A_2|$.

Let D be an acyclic orientation of $G - e$. If D has a path from x to y (for short, an "x,y-path") then it cannot have a y,x-path, so directing e as $\vec{xy}$ (but not $e = \vec{yx}$) produces an acyclic orientation of G; all this is true if we reverse the roles of x and y. We get every orientation in A_2 in this way. On the other hand, if D does not have either an x,y-path or a y,x-path, then we can orient e in either direction to produce an orientation in A_1. Therefore

$$a(G - e) \;=\; \frac{1}{2}|A_1| + |A_2|. \tag{5}$$

Now let D be an acyclic orientation of G/e, and let $\hat{D}$ be the corresponding acyclic orientation of $G - e$. I claim that $\hat{D}$ can be extended to an acyclic orientation of G by orienting e in either way. Indeed, if it were impossible to orient e as $\vec{xy}$, then the reason would have to be that $\hat{D}$ contained a path from y to x, but y and x are the same vertex in D and D wouldn't be acyclic. Therefore, there is a bijection between $A(G/e)$ and matched pairs $\{D, \tilde{D}\}$ in $A(G)$, so

$$a(G/e) \;=\; \frac{1}{2}|A_1|. \tag{6}$$

Now combining (5) and (6) proves **(A3)**. $\square$

Some other related graph-theoretic invariants you can find from the Tutte polynomial:

- The number of *totally cyclic orientations*, i.e., orientations in which every edge belongs to a directed cycle (HW problem).
- The *flow polynomial* of G, whose value at k is the number of edge-labelings $f : E \to [k-1]$ such that the sum at every vertex is 0 mod k.
- The *reliability polynomial* $f(p)$: the probability that the graph remains connected if each edge is deleted with independent probability p.
- The "enhanced chromatic polynomial", which enumerates all q-colorings by "improper edges":

$$\tilde{\chi}(q,t) = \sum_{f:V\to[q]} t^{\#\{xy\in E \ \mid \ f(x)=f(y)\}}.$$

  This is essentially Crapo's *coboundary polynomial*, and provides the same information as the Tutte polynomial.
- And more; the canonical source for all things Tutte is T. Brylawski and J. Oxley, "The Tutte polynomial and its applications," Chapter 6 of *Matroid applications*, N. White, editor (Cambridge Univ. Press, 1992).

Basis Activities

We know that $T(M; x, y)$ has nonnegative integer coefficients and that $T(M; 1, 1)$ is the number of bases of M. These observations suggest that we should be able to interpret the Tutte polynomial as a generating function for bases: that is, there should be combinatorially defined functions $i, e : \mathscr{B}(M) \to \mathbb{N}$ such that

$$T(M; x, y) \;=\; \sum_{B\in\mathscr{B}(M)} x^{i(B)} y^{e(B)}. \tag{7}$$

In fact, this is the case. The tricky part is that $i(B)$ and $e(B)$ must be defined with respect to a total order on the ground set E, so they are not really invariants of B itself. However, another miracle occurs: the right-hand side of (7) does not depend on this choice of total order.

Index the ground set of E as $\{e_1, \dots, e_n\}$, and totally order the elements of E by their subscripts.

Definition 1. Let B be a basis of M.

- Let $e_i \in B$. The **fundamental cocircuit** $C^*(e_i, B)$ is the unique cocircuit in $(E \setminus B) \cup e_i$. That is,

$$C^*(e_i, B) = \{e_j \mid B \setminus e_i \cup e_j \in \mathscr{B}\}.$$

  We say that e_i is **internally active** with respect to B if e_i is the minimal element of $C(e_i, B)$.

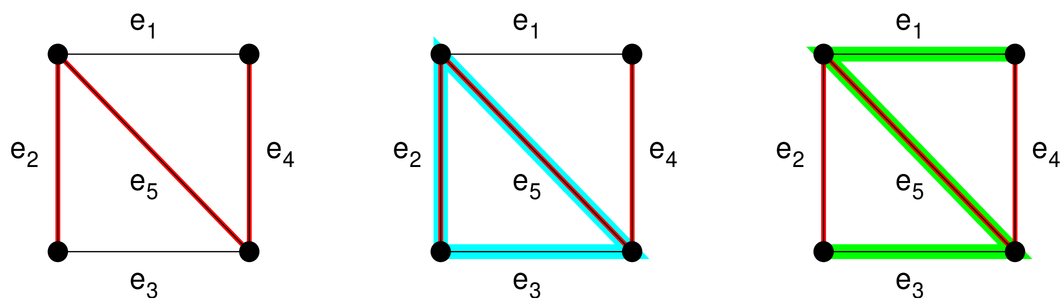
- Let $e_i \notin B$. The **fundamental circuit** $C(e_i, B)$ is the unique circuit in $B \cup e_i$. That is,

$$C(e_i, B) = \{e_j \mid B \setminus e_j \cup e_i \in \mathscr{B}\}.$$

  We say that e_i is **externally active** with respect to B if e_i is the minimal element of $C(e_i, B)$.

- Finally, we let $e(B)$ and $i(B)$ denote respectively the number of externally active and internally active elements with respect to B.

Here's an example. Let G be the graph with edges labeled as shown below, and let B be the spanning tree $\{e_2, e_4, e_5\}$ shown in red. The middle figure shows $C(e_1, B)$, and the right-hand figure shows $C^*(e_5, B)$.



Then

$C(e_1, B) = \{e_1, e_4, e_5\}$ so e_1 is externally active;

$C(e_3, B) = \{e_2, e_3, e_5\}$ so e_3 is not externally active.

Therefore $e(B) = 1$. Meanwhile,

$C^*(e_2, B) = \{e_2, e_3\}$ so e_1 is internally active;

$C^*(e_4, B) = \{e_1, e_4\}$ so e_3 is not internally active;

$C^*(e_5, B) = \{e_1, e_3, e_5\}$ so e_3 is not internally active.

Therefore $i(B) = 1$.

Theorem 3. *Let M be a matroid on E. Fix a total ordering of E and define $i, e : \mathscr{B}(M) \to \mathbb{N}$ as above. Then* (7) *holds.*

Thus, in the example above, the spanning tree B would contribute the monomial $xy = x^1y^1$ to $T(G; x, y)$.

The proof, which I'll omit, is just a matter of bookkeeping. It's a matter of showing that the generating function on the right-hand side of (7) satisfies the recursive definition of the Tutte polynomial.