Wednesday 2/20

The Tutte Polynomial

Definition 1. Let M be a matroid with ground set E and let $e \in E$. The **Tutte polynomial** $T(M) = T(M;\, x, y)$ is computed recursively as follows:

- **(T1)** If $E = \emptyset$, then $T(M) = 1$.
- **(T2a)** If $e \in E$ is a loop, then $T(M) = y \cdot T(M/e)$.
- **(T2b)** If $e \in E$ is a coloop, then $T(M) = x \cdot T(M - e)$.
- **(T3)** If $e \in E$ is neither a loop nor a coloop, then $T(M) = T(M - e) + T(M/e)$.

We prove that $T(M)$ is well-defined by giving a closed formula for it, the *corank-nullity** *generating function.*

Theorem 1. *Let r be the rank function of the matroid M. Then*

$$T(M;\, x, y) \;=\; \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \tag{1}$$

Proof. Let $\tilde{T}(M) = \tilde{T}(M;\, x, y)$ denote the generating function on the right-hand side of (1). We will prove by induction on $n = |E|$ that $\tilde{T}(M)$ obeys the recurrence of Definition 1 for every ground set element e, hence equals $T(M)$. Let r' and r'' denote the rank functions on $M - e$ and M/e respectively.

For **(T1)**, if $E = \emptyset$, then (1) gives $\tilde{T}(M) = 1 = T(M)$.

For **(T2a)**, let e be a loop. Then $r'(A) = r(A) = r(A \cup e)$ for every $A \subset E \setminus e$, so

$$\begin{aligned}
\tilde{T}(M) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} \\
&= \sum_{\substack{A \subseteq E \\ e \notin A}} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} + \sum_{\substack{B \subseteq E \\ e \in B}} (x-1)^{r(E)-r(B)} (y-1)^{|A|-r(B)} \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)} (y-1)^{|A|-r'(A)} + \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)} (y-1)^{|A|+1-r'(A)} \\
&= (1 + (y-1)) \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)} (y-1)^{|A|-r'(A)} \\
&= y \tilde{T}(M - e).
\end{aligned}$$

For **(T2b)**, let e be a coloop. Then $r''(A) = r(A) = r(A \cup e) - 1$ for every $A \subset E \setminus e$, so

$$\begin{aligned}
\tilde{T}(M) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} \\
&= \sum_{e \notin A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} + \sum_{e \in B \subseteq E} (x-1)^{r(E)-r(B)} (y-1)^{|A|-r(B)} \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{(r''(E \setminus e)+1)-r''(A)} (y-1)^{|A|-r''(A)} \\
&\quad + \sum_{A \subseteq E \setminus e} (x-1)^{(r''(E \setminus e)+1)-(r''(A)+1)} (y-1)^{|A|+1-(r''(A)+1)}
\end{aligned}$$

* The quantity $r(E) - r(A)$ is the *corank* of A; it is the minimum number of elements one needs to add to A to obtain a spanning set of M. Meanwhile, $|A| - r(A)$ is the *nullity* of A: the minimum number of elements one needs to remove from A to obtain an acyclic set.

$$
\begin{aligned}
&= \sum_{A\subseteq E\setminus e} (x-1)^{r''(E\setminus e)+1-r''(A)}(y-1)^{|A|-r''(A)} + \sum_{A\subseteq E\setminus e} (x-1)^{r''(E\setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= ((x-1)+1) \sum_{A\subseteq E\setminus e} (x-1)^{r''(E\setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= x\tilde{T}(M/e).
\end{aligned}
$$

Finally, suppose that e is neither a loop nor a coloop. Then

$$r'(A) = r(A) \quad \text{and} \quad r''(A) = r(A\cup e) - 1$$

so

$$
\begin{aligned}
\tilde{T}(M) &= \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\
&= \sum_{A\subseteq E\setminus e} \left[(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}\right] + \left[(x-1)^{r(E)-r(A\cup e)}(y-1)^{|A\cup e|-r(A\cup e)}\right] \\
&= \sum_{A\subseteq E\setminus e} \left[(x-1)^{r'(E\setminus e)-r'(A)}(y-1)^{|A|-r'(A)}\right] + \left[(x-1)^{(r''(E)+1)-(r''(A)+1)}(y-1)^{|A|+1-(r''(A)-1)}\right] \\
&= \sum_{A\subseteq E\setminus e} (x-1)^{r'(E\setminus e)-r'(A)}(y-1)^{|A|-r'(A)} + \sum_{A\subseteq E\setminus e} (x-1)^{r''(E\setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= \tilde{T}(M-e) + \tilde{T}(M/e)
\end{aligned}
$$

□

which is **(T3)**.

As a consequence, we can obtain several invariants of a matroid easily from its Tutte polynomial.

Corollary 2. *For every matroid M, we have*

1. $T(M;\,1,1)$ = *number of bases of M;*
2. $T(M;\,2,2)$ = $|E|$*;*
3. $T(M;\,2,1)$ = *number of independent sets of M;*
4. $T(M;\,1,2)$ = *number of spanning sets of M.*

Proof. We've already proved (1) and (2), but they also follow from the corank-nullity generating function. Plugging in $x = 2, y = 2$ will change every summand to 1. Plugging in $x = 1$ and $y = 1$ will change every summand to 0, *except* for those sets A that have corank and nullity both equal to 0 — that is, those sets that are both spanning and independent. The verifications of (3) and (4) are analogous. □

A little more generally, we can use the Tutte polynomial to enumerate independent and spanning sets by their cardinality:

$$\sum_{A\subseteq E \text{ independent}} q^{|A|} = q^{r(M)}T(1/q+1,1); \tag{2}$$

$$\sum_{A\subseteq E \text{ spanning}} q^{|A|} = q^{r(M)}T(1,1/q+1). \tag{3}$$

Another easy fact is that $T(M)$ is multiplicative on direct sums:

$$T(M_1 \oplus M_2) = T(M_1)T(M_2).$$

The Chromatic Polynomial

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a function $f : V \to [k]$; the coloring is *proper* if $f(v) \neq f(w)$ whenever verices v and w are adjacent. Let $\mathscr{X}_k(G)$ denote the set of proper colorings of G.

The function $k \mapsto |\mathscr{X}_k(G)|$ is called the *chromatic function* $\chi(G;\ k)$.

- If G has a loop, then its endpoints automatically have the same color, so $\chi(G;\ k) = 0$.
- If $G = K_n$, then all vertices must have different colors. There are k choices for $f(1)$, $k-1$ choices for $f(2)$, etc., so $\chi(K_n;\ k) = k(k-1)(k-2)\cdots(k-n+1)$.
- At the other extreme, let $G = \overline{K_n}$, the graph with n vertices and no edges. Then $\chi(\overline{K_n};\ k) = k^n$.
- If T_n is a tree with n vertices, then pick any vertex as the root; this imposes a partial order on the vertices in which the root is $\hat{1}$ and each non-root vertex v is covered by exactly one other vertex $p(v)$ (its "parent"). There are k choices for the color of the root, and once we know $f(p(v))$ there are $k-1$ choices for $f(v)$. Therefore $\chi(T_n;\ k) = k(k-1)^{n-1}$.
- $\chi(G + H;\ k) = \chi(G;\ k)\chi(H;\ k)$, where $+$ denotes disjoint union of graphs.

Theorem 3. *For every graph G we have*

$$\chi(G;\ k) \ = \ (-1)^{n(G)-1} k \cdot T(G,\ 1-k, 0)$$

where $n(G)$ is the number of vertices of G.