Monday 2/18

The Tutte Polynomial

Let M be a matroid with ground set E. Recall that we can *delete* or *contract* an element $e \in E$ to obtain respectively the matroids $M - e$ amd M/e on $E \setminus \{e\}$, whose basis systems are

$$\mathscr{B}(M-e) = \{B \mid B \in \mathscr{B}(M),\ e \notin B\},$$
$$\mathscr{B}(M/e) = \{B \setminus e \mid B \in \mathscr{B}(M),\ e \in B\}.$$

Thus deletion is defined whenever e is not a coloop, and contraction is defined whenever e is not a loop.

Definition 1. The **Tutte polynomial** of M is compute recursively as

$$T(M) = T(M;\, x, y) = \begin{cases} 1 & \text{if } E = \emptyset, \\ x \cdot T(M/e) & \text{if } e \text{ is a coloop}, \\ y \cdot T(M-e) & \text{if } e \text{ is a loop}, \\ T(M-e) + T(M/e) & \text{otherwise}, \end{cases} \tag{1}$$

for any $e \in E$.

If $M = M(G)$ is a graphic matroid, we may write $T(G)$ instead of $T(M(G))$.

This is more of an algorithm than a definition, and at this point, it is not even clear that $T(M)$ is well-defined, because the formula seems to depend on the order in which we pick elements to delete and contract. However, a miracle occurs: it doesn't! We will soon prove this by giving a closed formula for $T(M)$ that does not depend on any such choice.

In the case that M is a uniform matroid, then it *is* clear at this point that $T(M)$ is well-defined by (1), because, up to isomorphism, $M - e$ and M/e are independent of the choices of $e \in E$.

Example 1. Suppose that $M \cong U_n(n)$, that is, every element of E is a coloop. By induction, $T(M)(x, y) = x^n$. Dually, if $M \cong U_0(n)$ (i.e., every element of E is a loop), then $T(M)(x, y) = y^n$.

Example 2. Let $M \cong U_1(2)$ (the graphic matroid of the "digon", two vertices joined by two parallel edges). Let $e \in E$; then

$$\begin{aligned} T(M) &= T(M-e) + T(M/e) \\ &= T(U_1(1)) + T(U_0(1)) \ = \ x + y. \end{aligned}$$

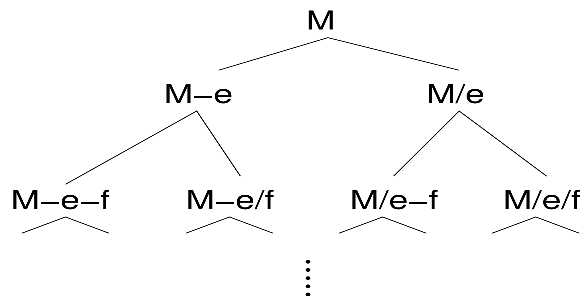
Example 3. Let $M \cong U_2(3)$ (the graphic matroid of K_3, as well as the matroid associated with the geometric lattice $\Pi_3 \cong M_5$). Applying (1) for any $e \in E$ gives

$$T(U_2(3)) \ = \ T(U_2(2)) + T(U_1(2)) \ = \ x^2 + x + y.$$

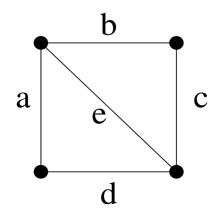
On the other hand,

$$T(U_1(3)) \ = \ T(U_1(2)) + T(U_0(2)) \ = \ x + y + y^2.$$

In general, we can represent a calculation of $T(M)$ by a binary tree in which moving down corresponds to deleting or contracting:



Example 4. Here is a non-uniform example. Let G be the graph below.



One possibility is to recurse on edge a (or equivalently on b, c, or d). When we delete a, the edge d becomes a coloop, and contracting it produces a copy of K_3. Therefore

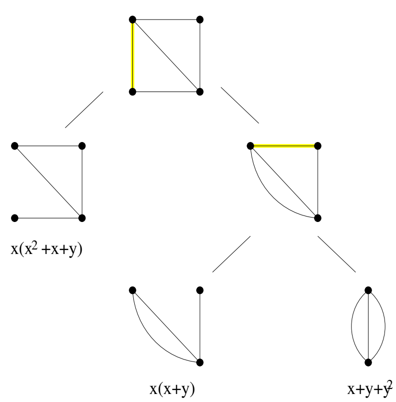
$$T(G-a) \;=\; x(x^2+x+y)$$

by Example 3. Next, apply the recurrence to the edge b in G/a. The graph $G/a-b$ has a coloop c, contracting which produces a digon. Meanwhile, $M(G/a/b) \cong U_1(3)$. Therefore

$$T(G/a-b) \;=\; x(x+y) \quad \text{and} \quad T(G/a/b) \;=\; x+y+y^2.$$

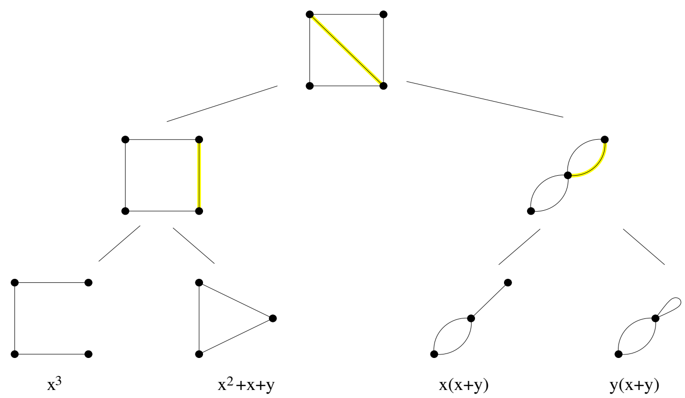
Putting it all together, we get

$$\begin{aligned} T(G) &= T(G-a)+T(G/a) \\ &= T(G-a)+T(G/a-b)+T(G/a/b) \\ &= x(x^2+x+y) \;+\; x(x+y) \;+\; (x+y+y^2) \\ &= x^3+2x^2+2xy+x+y+y^2. \end{aligned}$$



On the other hand, we could have recursed first on e, getting

$$\begin{aligned} T(G) &= T(G-e)+T(G/e) \\ &= T(G-e-c)+T(G-e/c)+T(G/e-c)+T(G/e/c) \\ &= x^3 \;+\; (x^2+x+y) \;+\; x(x+y) \;+\; y(x+y) \\ &= x^3+2x^2+2xy+x+y+y^2. \end{aligned}$$



We can actually see the usefulness of $T(M)$ even before proving that it is well-defined!

Proposition 1. *$T(M;\,1,1)$ equals the number of bases of M.*

Proof. Let $b(M) = T(M;\,1,1)$. Then (1) gives

$$b(M) = \begin{cases} 1 & \text{if } E = \emptyset, \\ b_{G/e} & \text{if } e \text{ is a coloop} \\ b_{G-e} & \text{if } e \text{ is a loop} \\ b(M-e) + b(M/e) & \text{otherwise} \end{cases}$$

which is identical to the recurrence for $|\mathscr{B}(M)|$ that we observed on Friday 2/15. □

Many other invariants of M can be found in this way by making appropriate substitutions for the indeterminates x, y in $T(M)$. In particular, if we let $c(M) = T(M;\,2,2)$, then

$$c(M) = \begin{cases} 1 & \text{if } E = \emptyset, \\ 2c_{G/e} & \text{if } e \text{ is a coloop} \\ 2c_{G-e} & \text{if } e \text{ is a loop} \\ c(M-e) + c(M/e) & \text{otherwise} \end{cases}$$

so $c(M) = 2^{|E|}$. This suggests that $T(M)$ ought to have a closed formula as a sum over subsets $A \subseteq E$, with each summand becoming 1 upon setting $x = 1$ and $y = 1$—for example, with each summand a product of powers of $x - 1$ and $y - 1$. In fact, this is the case.

Theorem 2. *Let r be the rank function of the matroid M. Then*

$$T(M;\,x,y) \;=\; \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{2}$$

The quantity $r(E) - r(A)$ is the *corank* of A; it is the minimum number of elements one needs to add to A to obtain a spanning set of M. Meanwhile, $|A| - r(A)$ is the *nullity* of A: the minimum number of elements one needs to remove from A to obtain an acyclic set. Accordingly, (2) is referred to as the the *corank-nullity generating function*.

(As an exercise, work out $T(G;\,x,y)$ for the graph G of Example 4; you shoudl get the same answer as above.)

Proof of Theorem 2. Let $\tilde{T}(M) = \tilde{T}(M;\,x,y)$ denote the generating function on the right-hand side of (2). We will prove by induction on $n = |E|$ that $\tilde{T}(M)$ obeys the recurrence (1) for every ground set element e, hence equals $T(M)$. Let r' and r'' denote the rank functions on $M - e$ and M/e respectively.

For the base case, if $E = \emptyset$, then (2) gives $\tilde{T}(M) = 1 = T(M)$.

If e is a loop, then $r'(A) = r(A) = r(A \cup e)$ for every $A \subset E \setminus e$, so

$$\begin{aligned}
\tilde{T}(M) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\
&= \sum_{\substack{A \subseteq E \\ e \notin A}} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} + \sum_{\substack{B \subseteq E \\ e \in B}} (x-1)^{r(E)-r(B)}(y-1)^{|A|-r(B)} \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)}(y-1)^{|A|-r'(A)} + \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)}(y-1)^{|A|+1-r'(A)} \\
&= (1 + (y-1)) \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)}(y-1)^{|A|-r'(A)} \\
&= y\tilde{T}(M-e).
\end{aligned}$$

If e is a coloop, then $r''(A) = r(A) = r(A \cup e) - 1$ for every $A \subset E \setminus e$, so

$$\begin{aligned}
\tilde{T}(M) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\
&= \sum_{e \notin A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} + \sum_{e \in B \subseteq E} (x-1)^{r(E)-r(B)}(y-1)^{|A|-r(B)} \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{(r''(E \setminus e)+1)-r''(A)}(y-1)^{|A|-r''(A)} \\
&\quad + \sum_{A \subseteq E \setminus e} (x-1)^{(r''(E \setminus e)+1)-(r''(A)+1)}(y-1)^{|A|+1-(r''(A)+1)} \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{r''(E \setminus e)+1-r''(A)}(y-1)^{|A|-r''(A)} + \sum_{A \subseteq E \setminus e} (x-1)^{r''(E \setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= ((x-1)+1) \sum_{A \subseteq E \setminus e} (x-1)^{r''(E \setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= x\tilde{T}(M/e).
\end{aligned}$$

Finally, suppose that e is neither a loop nor a coloop. Then

$$r'(A) = r(A) \quad \text{and} \quad r''(A) = r(A \cup e) - 1.$$

Therefore,

$$\begin{aligned}
\tilde{T}(M) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\
&= \sum_{A \subseteq E \setminus e} \left[(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}\right] + \left[(x-1)^{r(E)-r(A \cup e)}(y-1)^{|A \cup e|-r(A \cup e)}\right] \\
&= \sum_{A \subseteq E \setminus e} \left[(x-1)^{r'(E \setminus e)-r'(A)}(y-1)^{|A|-r'(A)}\right] + \left[(x-1)^{(r''(E)+1)-(r''(A)+1)}(y-1)^{|A|+1-(r''(A)-1)}\right] \\
&= \sum_{A \subseteq E \setminus e} (x-1)^{r'(E \setminus e)-r'(A)}(y-1)^{|A|-r'(A)} + + \sum_{A \subseteq E \setminus e} (x-1)^{r''(E \setminus e)-r''(A)}(y-1)^{|A|-r''(A)} \\
&= \tilde{T}(M-e) + \tilde{T}(M/e).
\end{aligned}$$

□