Monday 2/11

Graphic Matroids

Definition 1. Let G be a finite graph with vertices V and edges E. For each subset $A \subset E$, the corresponding *induced subgraph* of G is the graph $G|_A$ with vertices V and edges A. The *graphic matroid* or *complete connectivity matroid* $M(G)$ on E is defined by the closure operator

$$\begin{aligned}\bar{A} &= \{e = xy \in E \mid \ A \text{ contains a path from } x \text{ to } y\} \\ &= \{e = xy \in E \mid x, y \text{ belong to the same component of } G|_A\}.\end{aligned} \tag{1}$$

The associated rank function is

$$r(A) = \min\{|A'| : \ A' \subseteq A, \ \overline{A'} = \overline{A}\}.$$

Such a subset A' is called a *spanning forest* of A (or of $G|_A$).

Theorem 1. *Let $A' \subseteq A$. Then any two of the following conditions imply the third (and characterize spanning forests of A):*

(1) $r(A') = r(A)$;
(2) A' *is acyclic;*
(3) $|A'| = |V| - c$, *where c is the number of connected components of A.*

The flats of $M(G)$ correspond to the subgraphs whose components are all *induced subgraphs* of G. For $W \subseteq V$, the induced subgraph $G[W]$ is the graph with vertices W and edges $\{xy \in E \mid x, y \in W\}$.

Example 1. If G is a *forest* (a graph with no cycles), then no two vertices are joined by more than one path. Therefore, every edge set is a flat, and $M(G)$ is a Boolean algebra.

Example 2. If G is a cycle of length n, then every edge set of size $< n-1$ is a flat, but the closure of a set of size $n-1$ is the entire edge set. Therefore, $M(G) \cong U_{n-1}(n)$.

Example 3. If $G = K_n$ (the complete graph on n vertices), then a flat of $M(G)$ is the same thing as an equivalence relation on $[n]$. Therefore, $M(K_n)$ is naturally isomorphic to the partition lattice Π_n.

Equivalent Definitions of Matroids

In addition to rank functions, lattices of flats, and closure operators, there are many other equivalent ways to define a matroid on a finite ground set E. In the fundamental example of a linear matroid M, some of these definitions correspond to linear-algebraic notions such as linear independence and bases.

Definition 2. A **<u>(matroid) independence system</u>** $\mathscr{I}$ is a family of subsets of E such that

(2a) $\emptyset \in \mathscr{I}$;

(2b) if $I \in \mathscr{I}$ and $I' \subseteq I$, then $I' \in \mathscr{I}$; and

(2c) if $I, J \in \mathscr{I}$ and $|I| < |J|$, then there exists $x \in J \setminus I$ such that $I \cup x \in \mathscr{B}$.

<u>Note:</u> A family of subsets satisfying (2a) and (2b) is called a *simplicial complex* on E.

If E is a finite subset of a vector space, then the linearly independent subsets of E form a matroid independence system. Conditions (2a) and (2b) are clear. For condition (2c), the span of J has greater dimension than that of I, so there must be some $x \in J$ outside the span of I, and then $I \cup x$ is linearly independent.

A matroid independence system records the same combinatorial structure on E as a matroid rank function.

Proposition 2. *Let E be a finite set.*

(1) *If r is a matroid rank function on E, then*
$$\mathscr{I} = \{A \subset E \mid r(A) = |A|\}$$
is an independence system.
(2) *If $\mathscr{B}$ is an independence system on E, then*
$$r(A) = \max\{|I \cap A| \mid I \in \mathscr{B}\}$$
is a matroid rank function.
(3) *These constructions are mutual inverses.*

If $M = M(G)$ is a graphic matroid, the associated independence system is the family of *acyclic* edge sets in G. To see this, notice that if A is a set of edges and $e \in A$, then $r(A \setminus e) < r(A)$ if and only if deleting e breaks a component of $G|_A$ into two smaller components (so that in fact $r(A \setminus e) = r(A) - 1$. This is equivalent to the condition that e belongs to no cycle in A. Therefore, if A is acyclic, then deleting its edges one by one gets you down to $\emptyset$ and decrements the rank each time, so $r(A) = |A|$. On the other hand, if A contains a cycle, then deleting any of its edges won't change the rank, so $r(A) < |A|$.

Here's what the "donation" condition (2c) means in the graphic setting. Suppose that $|V| = n$, and let $c(H)$ denote the number of components of a graph H. If I, J are acyclic edge sets with $|I| < |J|$, then
$$c(G|_I) = n - |I| \;>\; c(G|_J) = n - |J|,$$
and there must be some edge $e \in J$ whose endpoints belong to different components of $G|_I$; that is, $I \cup e$ is acyclic.

The maximal independent sets are called *bases* of the matroid.

Definition 3. A **(matroid) basis system** $\mathscr{B}$ on E is a family of subsets of E such that, for all $B, B' \in \mathscr{B}$,

(3a) $|B| = |B'|$; and

(3b) for all $e \in B \setminus B'$, there exists $e' \in B' \setminus B$ such that $B \setminus e \cup e' \in \mathscr{B}$.

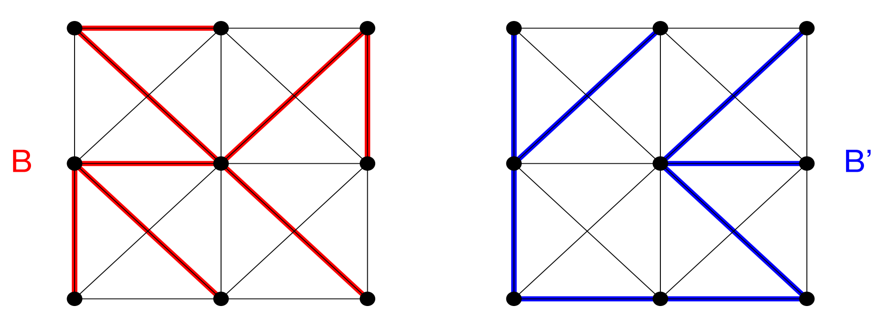
The condition (3b) can be replaced with

(3c) for all $e \in B \setminus B'$, there exists $e' \in B' \setminus B$ such that $B' \setminus e' \cup e \in \mathscr{B}$,

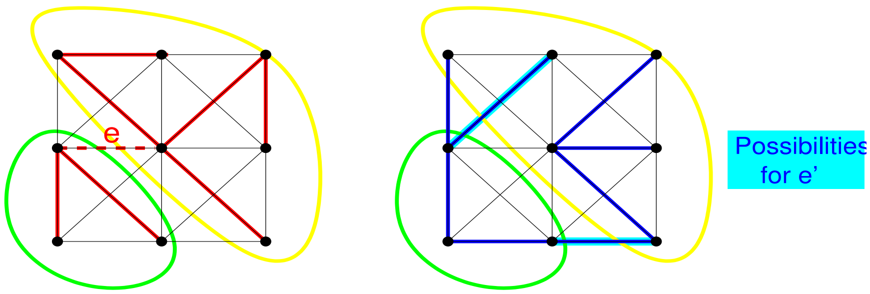
although this is not obvious (proof for homework).

Indeed, if S is a finite set of vectors spanning a vector space V, then the subsets of S that are bases for V all have the same cardinality (namely $\dim V$) and satisfy the basis exchange condition (3b).

If G is a connected graph, then the bases of $M(G)$ are its *spanning trees*.



Here's the interpretation of (3b). If $e \in B \setminus B'$, then $B \setminus e$ has two connected components. Since B' is connected, there must be some edge e' with one endpoint in each of those components, and then $B \setminus e \cup e'$ is a spanning tree.



As for (3c), if $e \in B \setminus B'$, then $B' \cup e$ must contain a unique cycle C (formed by e together with the unique path in B' between the endpoints of e). Deleting any edge $e' \in C \setminus e$ will produce a spanning tree.