Wednesday 1/30/08

**Modular Lattices**

**Definition:** A lattice $L$ is **modular** if for every $x, y, z \in L$ with $x \leq z$,

\[ x \lor (y \land z) = (x \lor y) \land z. \]

(Note: For all lattices, if $x \leq z$, then $x \lor (y \land z) \leq (x \lor y) \land z$.)

**Some basic facts and examples:**

1. Every sublattice of a modular lattice is modular. Also, if $L$ is distributive and $x \leq z \in L$, then

\[ x \lor (y \land z) = (x \land z) \lor (y \land z) = (x \lor y) \land z, \]

so $L$ is modular.

2. $L$ is modular if and only if $L^*$ is modular. Unlike the corresponding statement for distributivity, this is completely trivial, because the definition of modularity is invariant under dualization.

3. $N_5$ is not modular. With the labeling below, we have $a \leq b$, but

\[ a \lor (c \land b) = a \lor \hat{0} = a, \]

\[ (a \lor c) \land b = \hat{1} \land b = b. \]

4. $M_5 \cong \Pi_3$ is modular. However, $\Pi_4$ is not modular (exercise).

Modular lattices tend to come up in algebraic settings:

- Subspaces of a vector space
- Subgroups of a group
- $R$-submodules of an $R$-module

E.g., if $X, Y, Z$ are subspaces of a vector space $V$ with $X \subseteq Z$, then the modularity condition says that

\[ X + (Y \cap Z) = (X + Y) \cap Z. \]

**Proposition 1.** Let $L$ be a lattice. TFAE:

1. $L$ is modular.
2. For all $x, y, z \in L$, if $x \in [y \land z, z]$, then $x = (x \lor y) \land z$.

2*. For all $x, y, z \in L$, if $x \in [y, y \lor z]$, then $x = (x \land z) \lor y$.

3. For all $y, z \in L$, there is an isomorphism of lattices

\[ [y \land z, z] \cong [y, y \lor z] \]

given by $a \mapsto a \lor y, b \land z \mapsto b$. 

Proof. (1) $\implies$ (2) is easy: if we take the definition of modularity and assume in addition that $x \geq y \land z$, then the equation becomes $x = (x \lor y) \land z$.

For (2) $\implies$ (1), suppose that (2) holds. Let $X, Y, Z \in L$ with $X \leq Z$. Note that

$$Y \land Z \leq X \lor (Y \land Z) \leq Z \lor Z = Z,$$

so applying (2) with $y = Y$, $z = Z$, $x = X \lor (Y \land Z)$ gives

$$X \lor (Y \land Z) = ((X \lor (Y \land Z)) \lor Y) \land Z = (X \lor Y) \land Z$$

as desired.

(2) $\iff$ (2*) because modularity is a self-dual condition.

Finally, (3) is equivalent to (2) and (2*) together.

Theorem 2. Let $L$ be a lattice.

1. $L$ is modular if and only if it contains no sublattice isomorphic to $N_5$.
2. $L$ is distributive if and only if it contains no sublattice isomorphic to $N_5$ or $M_5$.

Proof. Both $\implies$ directions are easy, because $N_5$ is not modular and $M_5$ is not distributive.

Suppose that $x, y, z$ is a triple for which modularity fails. One can check that

is a sublattice (details left to the reader).

Suppose that $L$ is not distributive. If it isn’t modular then it contains an $N_5$, so there’s nothing to prove. If it is modular, then choose $x, y, z$ such that

$$x \land (y \lor z) > (x \land y) \lor (x \land z).$$

You can then show that

1. this inequality is invariant under permuting $x, y, z$;
2. $[x \land (y \lor z)] \lor (y \land z)$ and the two other lattice elements obtained by permuting $x, y, z$ form a cochain; and
3. the join (resp. meet) of any of two of those three guys is equal.

Hence, we have constructed a sublattice of $L$ isomorphic to $M_5$. 

\qed
**Semimodular Lattices**

**Definition:** A lattice $L$ is **(upper) semimodular** if for all $x, y \in L$,

\[(2) \quad x \wedge y \leq y \implies x \leq x \vee y.\]

Here’s the idea. Consider the interval $[x \wedge y, x \vee y] \subset L$.

![Diagram](image)

If $L$ is semimodular, then the interval has the property that if the southeast relation is a cover, then so is the northwest relation.

$L$ is **lower semimodular** if the converse of (2) holds for all $x, y \in L$.

**Lemma 3.** If $L$ is modular then it is upper and lower semimodular.

**Proof.** If $x \wedge y \leq y$, then the sublattice $[x \wedge y, y]$ has only two elements. If $L$ is modular, then by condition (3) of Proposition we have $[x \wedge y, y] \cong [x, x \vee y]$, so $x \leq x \vee y$. Hence $L$ is upper semimodular. A similar argument proves that $L$ is lower semimodular. \(\square\)

In fact, upper and lower semimodularity together imply modularity. To make this more explicit, we will show that each of these three conditions on a lattice $L$ implies that it is ranked, and moreover, for all $x, y \in L$, the rank function $r$ satisfies

- $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ if $L$ is upper semimodular;
- $r(x \vee y) + r(x \wedge y) \geq r(x) + r(y)$ if $L$ is lower semimodular;
- $r(x \vee y) + r(x \wedge y) = r(x) + r(y)$ if $L$ is modular.