Wednesday 1/30/08

Modular Lattices

Definition: A lattice L is **modular** if for every $x, y, z \in L$ with $x \leq z$,

(1) $$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

(Note: For all lattices, if $x \leq z$, then $x \vee (y \wedge z) \leq (x \vee y) \wedge z$.)

Some basic facts and examples:

1. Every sublattice of a modular lattice is modular. Also, if L is distributive and $x \leq z \in L$, then

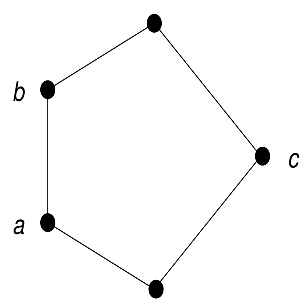
$$x \vee (y \wedge z) \;=\; (x \wedge z) \vee (y \wedge z) \;=\; (x \vee y) \wedge z,$$

so L is modular.

2. L is modular if and only if L^* is modular. Unlike the corresponding statement for distributivity, this is completely trivial, because the definition of modularity is invariant under dualization.

3. N_5 is not modular. With the labeling below, we have $a \leq b$, but

$$a \vee (c \wedge b) = a \vee \hat{0} \;=\; a,$$
$$(a \vee c) \wedge b = \hat{1} \wedge b \;=\; b.$$



4. $M_5 \cong \Pi_3$ is modular. However, Π_4 is not modular (exercise).

Modular lattices tend to come up in algebraic settings:

- Subspaces of a vector space
- Subgroups of a group
- R-submodules of an R-module

E.g., if X, Y, Z are subspaces of a vector space V with $X \subseteq Z$, then the modularity condition says that

$$X + (Y \cap Z) = (X + Y) \cap Z.$$

Proposition 1. *Let L be a lattice. TFAE:*

1. *L is modular.*
2. *For all $x, y, z \in L$, if $x \in [y \wedge z, z]$, then $x = (x \vee y) \wedge z$.*

2^*. *For all $x, y, z \in L$, if $x \in [y, y \vee z]$, then $x = (x \wedge z) \vee y$.*

3. *For all $y, z \in L$, there is an isomorphism of lattices*

$$[y \wedge z, z] \rightarrow [y, y \vee z]$$

given by $a \mapsto a \vee y$, $b \wedge z \leftarrow b$.

Proof. (1) $\implies$ (2) is easy: if we take the definition of modularity and assume in addition that $x \geq y \wedge z$, then the equation becomes $x = (x \vee y) \wedge z$.

For (2) $\implies$ (1), suppose that (2) holds. Let $X, Y, Z \in L$ with $X \leq Z$. Note that

$$Y \wedge Z \leq X \vee (Y \wedge Z) \leq Z \vee Z = Z,$$

so applying (2) with $y = Y$, $z = Z$, $x = X \vee (Y \wedge Z)$ gives

$$X \vee (Y \wedge Z) \;=\; \big((X \vee (Y \wedge Z)) \vee Y\big) \wedge Z \;=\; (X \vee Y) \wedge Z$$

as desired.

(2) $\iff$ (2*) because modularity is a self-dual condition.

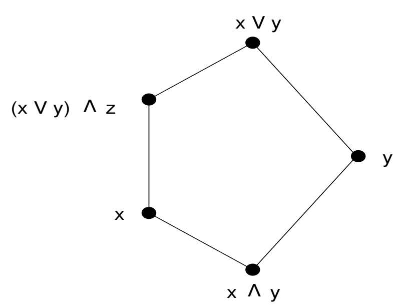
Finally, (3) is equivalent to (2) and (2*) together. □

Theorem 2. *Let L be a lattice.*

(1) *L is modular if and only if it contains no sublattice isomorphic to N_5.*
(2) *L is distributive if and only if it contains no sublattice isomorphic to N_5 or M_5.*

Proof. Both $\implies$ directions are easy, because N_5 is not modular and M_5 is not distributive.

Suppose that x, y, z is a triple for which modularity fails. One can check that



is a sublattice (details left to the reader).

Suppose that L is not distributive. If it isn't modular then it contains an N_5, so there's nothing to prove. If it is modular, then choose x, y, z such that

$$x \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z).$$

You can then show that

(1) this inequality is invariant under permuting x, y, z;
(2) $[x \wedge (y \vee z)] \vee (y \wedge z)$ and the two other lattice elements obtained by permuting x, y, z form a cochain; and
(3) the join (resp. meet) of any of two of those three guys is equal.

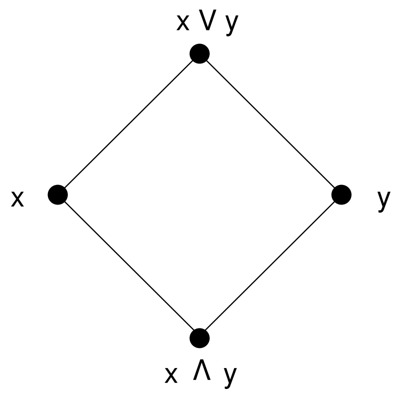
Hence, we have constructed a sublattice of L isomorphic to M_5. □

Semimodular Lattices

Definition: A lattice L is **(upper) semimodular** if for all $x, y \in L$,

$$x \wedge y \lessdot y \implies x \lessdot x \vee y. \tag{2}$$

Here's the idea. Consider the interval $[x \wedge y, x \vee y] \subset L$.



If L is semimodular, then the interval has the property that if the southeast relation is a cover, then so is the northwest relation.

L is **lower semimodular** if the converse of (2) holds for all $x, y \in L$.

Lemma 3. *If L is modular then it is upper and lower semimodular.*

Proof. If $x \wedge y \lessdot y$, then the sublattice $[x \wedge y, y]$ has only two elements. If L is modular, then by condition (3) of Proposition 1 we have $[x \wedge y, y] \cong [x, x \vee y]$, so $x \lessdot x \vee y$. Hence L is upper semimodular. A similar argument proves that L is lower smimodular. □

In fact, upper and lower semimodularity together imply modularity. To make this more explicit, we will show that each of these three conditions on a lattice L implies that it is ranked, and moreover, for all $x, y \in L$, the rank function r satisfies

$$\begin{aligned}
r(x \vee y) + r(x \wedge y) &\leq r(x) + r(y) && \text{if } L \text{ is upper semimodular;} \\
r(x \vee y) + r(x \wedge y) &\geq r(x) + r(y) && \text{if } L \text{ is lower semimodular;} \\
r(x \vee y) + r(x \wedge y) &= r(x) + r(y) && \text{if } L \text{ is modular.}
\end{aligned}$$