Math 724, Fall 2017
Take-Home Test #2
Deadline: Monday, November 13, 5:00pm

Instructions: Typeset your solutions in LaTeX. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name, e.g., `HilbertTest2.pdf`. You may refer to the textbook and your class notes, and you may cite the result of any problem from Chapter 1 assigned on Homeworks #1–5 or done in class. You may also use a computer algebra system such as Sage to carry out calculations and test conjectures. However, *you are not allowed to collaborate*; you may not consult any external resource or any human other than Jeremy.

Problem #1 Let $n > 0$ be an integer.

(#1a) [10 pts] How many labeled trees T on vertex set $[n]$ have the property that the degree of every vertex is either 1 or 3? Your answer should be a function of n expressed without summation notation.

(#1b) [10 pts] Let $L \subseteq [n]$ and let $k = |L|$. How many labeled trees T on vertex set $[n]$ have the property that every element of L is a leaf of T? (T can have other leaves as well.) Your answer should be a function of n and k expressed without summation notation.

Problem #2 [20 pts] Let $S(k, n)$ denote Stirling numbers of the second kind. Give a combinatorial proof that

$$S(k,n) = \sum_{i=1}^{k} \binom{k-1}{i-1} S(k-i, n-1)$$

for all positive integers k, n. (By "combinatorial," I mean "explain why both sides of the equation count the same set of objects" — do not give a purely algebraic proof using, say, induction.)

Problem #3 Give combinatorial interpretations for the following numbers (i.e., describe what they count).

(#3a) [10 pts] The coefficient of x^k in the infinite product

$$\prod_{n=1}^{\infty} (1 + x^n + x^{2n} + \cdots + x^{n^2}).$$

(#3b) [10 pts] The coefficient of x^k in the infinite product

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^{n^2}}.$$

Problem #4 [20 pts] Let k, n be positive integers and let $P(k, n)$ denote the number of partitions of k into n parts (as in Bogart, p. 70). Give a combinatorial proof that

$$P(k,1) + \cdots + P(k,n) = P(n+k,n).$$

Problem #5 [20 pts] Recall that 1 Galleon is worth 17 Sickles and 1 Sickle is worth 29 Knuts. Suppose that the Ministry introduces a 3-Sickle and a 6-Knut piece (known respectively as a Trickle and a Hexknut). With the new coinage, how many ways are there of making change for a Galleon? (If you are not an expert at Arithmancy, I recommend that you use Sage or another computer algebra system to do the calculation; if so, include the code you executed.)