Math 724, Fall 2013
Take-Home Test #1

Instructions: Write up your solutions using LaTeX. You may use books and notes, but you are not allowed to collaborate — you may not consult any human other than the instructor. Solutions are due at the start of class on **Friday, September 13.**

Problem #1 In a bridge deal, each of 4 players (North, South, West and East) is dealt a hand of 13 cards from a standard deck of 52 cards.

(#1a) [5 pts] How many bridge hands contain exactly four spades?

(#1b) [10 pts] How many bridge hands contain more spades than hearts?

(#1c) [5 pts] How many different possible deals are there?

Problem #2 [10 pts] Recall from Supplementary Problem 1 that a composition is an expression $n = a_1 + \cdots + a_k$, where the a_i are positive integers. A *weak composition* is the same thing, except that the a_i's are only required to be nonnegative feather than positive. Count the weak compositions of n into k parts.

Problem #3 [20 pts] Call a 3-digit number (in base 10) *purple* if its digits are in strictly increasing order. E.g., 314 and 288 are not purple, but 159 is purple. In order to solve the problem, give a bijection between the set of purple 3-digit numbers and something easily counted. You do not have to spend a lot of time proving that the bijection you construct is a bijection.

Problem #4 [20 pts] Let n be an integer not divisible by 2 or 5. Prove that there is some multiple of n whose decimal expansion consists of all 9's.

Problem #5 [10 pts] A *standard tableau of shape* $2 \times n$ is a $2 \times n$ grid filled with the numbers $1, \ldots, 2n$, using each number once, so that every row increases left to right and every column increases top to bottom. For example, there are five standard tableaux of shape 2×3:

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

Prove that for all $n > 0$, the number of standard tableaux of shape $2 \times n$ is the Catalan number C_n.

Problem #6 [20 pts] Let $m \geq n \geq 0$ be integers. Let $C(m, n)$ be the number of lattice paths from $(0, 0)$ to (m, n) that do not go above the line $y = x$. (So if $m = n$, then $C(m, n)$ is just the Catalan number C_n. Find a simple formula for $C(m, n)$ that generalizes the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. (Hint: Generalize the method of problem 51 in the textbook.)