Math 724, Fall 2013
Homework #5

Instructions: Write up your solutions in LaTeX and hand in a hard copy in class on **Friday, November 8.** Collaboration is allowed (and in fact encouraged), but each student must write up his or her solutions independently and acknowledge all collaborators.

(#1) Problem #168.

(#2) Problem #170.

(#3) Problem #181.

(#4) Problem #195.

(#5) Problem #196.

(#6) Problem #197.

(#7) Problem #205.

(#8) Problem #206.

(#9) Problem #207.

(Extra credit) Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring[1] of polynomials in n variables with complex coefficients. As a vector space, R decomposes as a direct sum

$$R = \bigoplus_{k=0}^{\infty} R_k = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_k means the vector subspace of polynomials that are homogeneous of degree k. (This "direct sum" statement is just a fancy way of saying that every polynomial can be written in exactly one way as a constant plus a homogeneous linear form plus a homogeneous quadratic form plus yada yada yada.) The *Hilbert series* of R is defined as the generating function

$$H_R(q) = \sum_{k \geq 0} (\dim R_k) q^k.$$

Finally, here's the problem: Find a nice closed-form expression for $H_R(q)$.

(Note: The Hilbert series can be defined for every graded ring, i.e., every ring R with a decomposition $R = \bigoplus_k R_k$ such that $R_k R_\ell \subseteq R_{k+\ell}$. Graded rings are ubiquitous in algebraic combinatorics, and the Hilbert series of a graded ring is its most fundamental invariant. For example, the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by any homogeneous ideal (i.e., any ideal I generated by homogeneous polynomials) is a graded ring; one remarkable fact is that the Hilbert series of $\mathbb{C}[x_1, \ldots, x_n]/I$ is always a rational function of q.)

[1] Quick definition of "ring": a set in which addition, subtraction and multiplication are well-defined operations that behave the way you think they ought to.