Math 724, Fall 2013
Homework #3

Instructions: Write up your solutions in LaTeX and hand in a hard copy in class on **Friday, October 4.** Collaboration is allowed (and in fact encouraged), but each student must write up his or her solutions independently and acknowledge all collaborators.

(#1) Problem #82.

(#2) Problem #83.

(#3) Problem #91.

(#4) Problem #116.

(#5) Let $G = (V, E)$ be a simple graph with n vertices ($n \geq 2$). ("Simple" means that every edge has two different vertices as its endpoints, and no two edges have the same pair of endpoints. So E can be regarded as a set of two-element subsets of V.) Recall that the *degree* of vertex i, written $d_G(i)$, is the number of neighbors of i.

Prove that G has two vertices with the same degree.

(#6) Let $G = (V, E)$ be a simple graph. Prove that the following conditions are equivalent. (Some of the implications should follow from problems done in class, but you should still give complete proofs – i.e., don't cite problems from Bogart in your answers.)

(1) G is connected and acyclic.
(2) G is connected and $|E| = |V| - 1$.
(3) G is acyclic and $|E| = |V| - 1$.
(4) Every two vertices in G are joined by exactly one path.

(#7) Let $G = (V, E)$ be a connected simple graph, with vertex set $V = \{v_1, \ldots, v_n\}$. The *Laplacian matrix* of G is the $n \times n$ square matrix $L(G) = [\ell_{ij}]_{i,j=1}^{n}$, with entries

$$\ell_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

The *reduced Laplacian* $L^i(G)$ is the $(n-1) \times (n-1)$ matrix obtained from $L(G)$ by crossing out the i^{th} row and column.

(a) Prove that for a given graph G, the number $\det L^i(G)$ is independent of the choice of i. (This part of the problem is really about linear algebra rather than combinatorics.) Henceforth, call that number $\tau(G)$.
(b) Calculate $\tau(G)$ for several different graphs, including trees, cycles, and complete graphs. You can use Sage. What do you notice? Can you prove your conjecture in any special cases?

Extra credit: Problem #86.