Math 410, Spring 2013
Homework Problems (updated 5/1/13)

In the 6th century, the Indian mathematician Aryabhata wrote that "*Half the circumference multiplied by half the diameter is the area of a circle.*"

Problem #1 Is Aryabhata's statement correct? Why or why not?

The quotation above doesn't say anything about the actual numerical value of π. However, Aryabhata also gave the following rule.

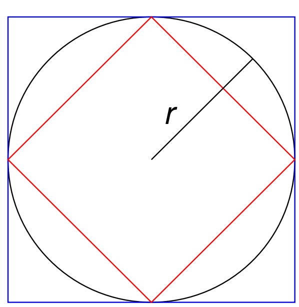
> *Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.*

Problem #2 What numerical value of π is implicit in Aryabhata's formula?

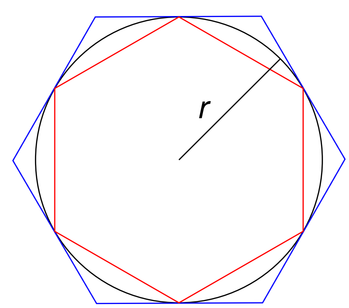
In class, we (briefly) discussed a method for estimating the value of π, which I attributed to Archimedes; actually, Archimedes thought of π as the ratio of circumference to diameter, not area to radius-squared. What Archimedes' method boils down to is this: if you inscribe a regular polygon with n sides inside a circle, then its perimeter, A_n, will be a bit more than the circumference of the circle. Likewise, if you circumscribe a regular polygon around a circle, then its perimeter B_n will be a bit less than the circumference. That is,

$$A_n \;>\; \text{circumference} \;>\; B_n.$$

The more sides the polygon has, the closer these estimates will be (and, of course, the harder to calculate).



perimeter of large square: A_4
perimeter of small square: B_4



perimeter of large hexagon: A_6
perimeter of small hexagon: B_6

Problem #3 Figure out the formulas for A_n and B_n, in terms of r for $n = 4$ and $n = 6$ (i.e., when the regular polygon is a square or a hexagon, as in the above figure). Decimal approximations are perfectly OK. (Archimedes did it by hand for a 96-sided polygon — an amazing feat considering trigonometry, not to mention calculators, hadn't been invented yet!)

Extra credit: Who first realized that $\pi = A/r^2$ and $\pi = C/(2r)$ were the same number? I don't know the answer, but I'm curious. Poke around the Internet and tell me what you find about this question.

Problem #4 Read the blog post Babylon and the Square Root of 2 and give a short explanation of how the Babylonians might have come up with their excellent approximation

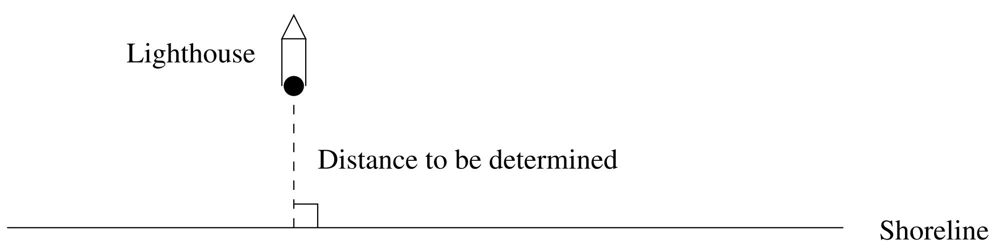
$$\sqrt{2} \approx 1 + 24/60 + 51/60^2 + 10/60^3.$$

If it's clear they read the blog post, 4/5. If they in addition understand something about it, 5/5.

Problem #5 There is an alternate proof that $\sqrt{2}$ is irrational. As before, we suppose that $\sqrt{2}$ can be written as a fraction P/Q, where P and Q are positive integers. However, let's make an additional assumption that the fraction P/Q is in lowest terms (i.e., P and Q have no common factors).[1]

Complete the proof by following the logic of the Pythagorean proof presented in class (also see the class notes) and finding a contradiction. (Hint: This assumption frees you from having to worry about infinite sequences.)

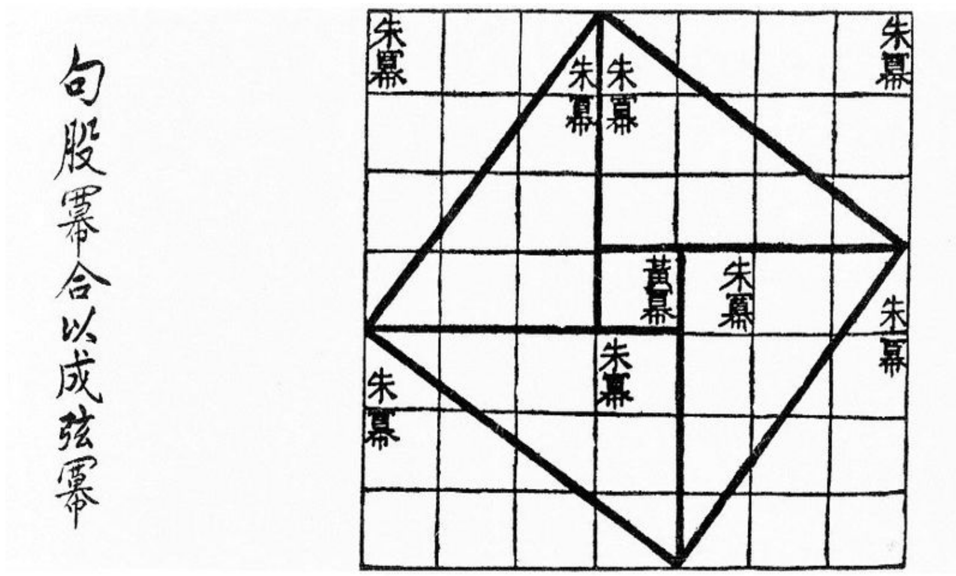
Problem #6 The year is 500 BCE. You are a Greek mathematician, currently working for the Egyptian government as a consultant. You've been asked to determine the distance from a lighthouse, which stands on a rock in the sea, to the mainland (see figure).



Explain to your Egyptian employers how you're going to carry out the project, and why your method works. You can use your compass, and you can measure as many line segments as you need, as long as they are all between points on land (since the slaves who are going to carry out the measurements can't swim). You can't use a protractor since they haven't been invented yet.

[1]The statement "every fraction can be written in lowest terms" itself has to be proved — but let's not worry about that for now.

What we call the Pythagorean theorem may have been known in some form to the Egyptians and Babylonians, and was definitely known (as *Gougu*) to ancient Chinese mathematicians. An early and elegant proof of Gougu comes from the *Zhoubi suanjing (Zhou Shadow Gauge Manual)*, which was compiled sometime between 100 BCE and 100 CE. (Pythagoras lived around 500 BCE, but the original sources for the Zhoubi suanjing may be as early as 1100 BCE — so it is unclear, and probably moot, who was first.) The text contains the following diagram:[2]



Problem #7 Using this diagram, explain why each side of the big square has length 5 (making it the hypotenuse of a 3-4-5 right triangle). Don't use the Pythagorean theorem in your answer (since that is ultimately what you're trying to explain). Hint: Find the area of the big square.

The *Zhoubi suanjing* contains the following statement:[3]

> A person gains knowledge by analogy, that is, after understanding a particular line of argument they can infer various kinds of similar reasoning.... Whoever can draw inferences about other cases from one instance can generalize.... To be able to deduce and then generalize... is the mark of an intelligent person.

Problem #8 In that spirit, take your explanation in the previous problem and use it to explain why the Pythagorean theorem works in general: that is, for any right triangle with sides of lengths a and b and hypotenuse of length c, the equation $a^2 + b^2 = c^2$ holds.

[2]Retrieved from Wikimedia Commons on 2/19/13; in the public domain.
[3]Taken from the MacTutor overview of Chinese mathematics.

Problem #9 While it has been known since Euclid that there are infinitely many primes (and there are several other ways of proving it, though none as elementary as Euler's), there are still lots and lots of open problems about prime numbers that remain mysterious. Two problems we discussed in class are the Twin Primes Conjecture ("there are infinitely many pairs of primes whose difference is 2") and the Goldbach Conjecture ("every positive number other than 2 can be written as the sum of two odd primes"). On the other hand, there are several refinements of Euclid's theorem, at least one of them proved within the last ten years.

Get on the Internet and find out for yourself (and tell me in a sentence or two) what the following things are: (a) Dirichlet's theorem on primes; (b) the Prime Number Theorem; (c) the Green-Tao Theorem.

It was pointed out in class that you can test whether a number n is divisible by 3 by adding up its digits. For example, if you add up the digits of $n = 72465702$, you get $7+2+4+6+5+7+0+2 = 33$. Since 33 is a multiple of 3, so is 72465702. The same test works for divisibility by 9 — since 33 is not a multiple of 9, neither is 72465702.

Here's something else you might try doing to a number n: look at the *alternating* sum of the digits. That is, add up the digits, but put a minus sign on every other digit. For example,

$$n = 72465702 \quad \text{becomes} \quad 7-2+4-6+5-7+0-2 \;=\; 5.$$

(Of course, it's possible to get zero or a negative number in this way, but so what?)

Miracle of miracles! This procedure really is a test for divisibility by some number d. Your job is to figure out what d is.

Problem #10 (i) Suppose that you apply this test to all positive numbers n from 1 to 100. (Don't actually do it — just think about what would happen.) What are all the values of n for which the alternating sum of digits is 0?

(ii) Based on your answer to (i), make a conjecture about the value of d.

(iii) Test your conjecture on these numbers: 2009, 3124, 4567, 8481, 218702088. (That is, for each of those numbers, form the alternating sum and say what your conjecture implies about whether it is divisible by d or not. Then check your answer by actually dividing the number by d — it is OK to use a calculator for this step.)

In class, we talked about an alternate method to find the circumference of the earth. Lie down on the ground and watch the sun set. (Remember, the sun isn't really setting — the earth is rotating on its axis so that the sun is disappearing from view.) At the instant the sun dips below the horizon, stand up straight so that the sun becomes visible again. Start a stopwatch and measure the time T it takes the sun to disappear.

(A more practical way to measure T might be to climb a tall tower of height H and have a friend stand at its base; each of you writes down the time when the sun disappears from view and then you subtract.)

Problem #11 Explain how to determine the radius of the earth in terms of the time T. Include a relevant figure, with all necessary parts of it labeled.

Full disclosure: I have never seen this before, so you really do have to convince me that your argument is correct!

Relevant observations made in class:

- The time gap T should be proportional to the angle through which the earth has rotated in that time.
- At the instant of sunset, your sight line to the sun should be tangent to the earth's surface.
- The final answer is independent of both T and H. On the other hand, both of them will show up in the formula. Note that T depends on H — the bigger H is, the bigger T will be.

As a historical note, this technique might not have been available to the ancient Greeks, since it requires a reasonably accurate clock and knowing how long the day is (it's not exactly 24 hours). Also, I do not know if they knew that the earth rotates on its axis. On the other hand, it's plausible that combining this idea with a previous estimate of the earth's radius or circumference — say, the one Eratosthenes came up with — could be a way of measuring how long a day is. (I don't know if this is historical either.)

Problem #12 Verify that Archimedes' trisection of the angle (described in the class notes) is correct.

Problem #13 Here is a Euclidean construction that comes close to being a trisection, but doesn't quite work. (Many people have discovered this construction and thought that it worked. In fact, lots and lots of people throughout history have insisted wrongly that they were able to solve the classical trisection problem, by this or other Euclidean constructions.)

1. Let $\angle AOB$ be the angle to be trisected.
2. Construct circles of radii 2, 3, and 4 centered at O. Call them C_2, C_3 and C_4 respectively.
3. Bisect $\angle BOA$ with a line ℓ. Let C be the point where line ℓ meets C_2.
4. Bisect $\angle COB$ with a line m. Let D be the point where line m meets C_4.
5. Draw the segment $\overline{CD}$. Let T be the point where $\overline{CD}$ meets C_3.
6. Construct $\angle TOB$. (This is supposed to be the trisection of $\angle AOB$.)

Carry out this construction in Sketchpad. (You don't have to send me the sketch.)

a. By moving A around, tell me the value of $m\angle TOB$ for each of these values of $\angle AOB$: $30°$, $90°$, $144°$, $6°$, $180°$.

b. Tell me how the value of $\angle AOB$ (i.e., small, medium or large) affects the accuracy of the construction.

Problem #14 **(Extra credit)** Prove that $\sqrt{n}$ is constructible for every odd positive integer n.

Problem #15 The Babylonian method of approximating $\sqrt{r}$ was to start with an estimate s_0, then to compute successively more accurate approximations $s_1, s_2, s_3 \ldots$ using the formula

$$s_{n+1} = \frac{s_n + r/s_n}{2}.$$

a. Use the Babylonian method to approximate $\sqrt{3}$, using a starting estimate of $s_0 = 1$. How many iterations do you need for s_n to be accurate to 7 decimal places?

b. What happens if you start with a less accurate estimate, such as $s_0 = 5$?

c. What happens if you start with $s_0 = -1$? (The Babylonians probably wouldn't have tried this because, so far as we know, they didn't know about negative numbers.)

d. What happens if you try to approximate $\sqrt{-3}$ by using a starting estimate of $s_0 = -1$? (The Babylonians *certainly* wouldn't have tried this.)

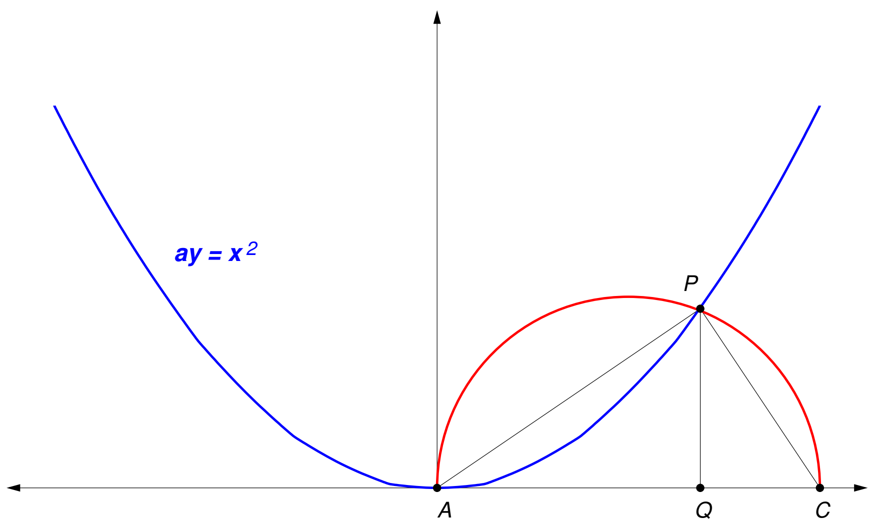
Problem #16 (Adapted from Burton, pp. 300–301) The Persian Omar Khayyam (c. 1050–1130 CE), best known as a poet, was also an outstanding mathematician. Several centuries before del Ferro (and/or Tartaglia and/or Cardano) solved the cubic equation algebraically, Khayyam came up with a geometric solution. His solution is non-Euclidean because it involves a parabola, but it's not hard to see that it works.

Khayyam considered the equation

$$z^3 + a^2 z = b$$

where a and b are positive real numbers. His solution is as follows (in modern coordinate notation):

1. Construct the parabola with equation $x^2 = ay$ (shown in blue below).
2. Construct a semicircle with diameter $AC = b/a^2$ on the x-axis (shown in red below).
3. Let P be the point where the parabola meets the semicircle. Drop a perpendicular from P to the x-axis to find the point Q.
4. Let z be the length of segment AQ.



Verify that Khayyam's construction of z is correct, by the following steps.

a. Prove that

$$z^2 = a \cdot PQ.$$

b. Prove that

$$\frac{z}{PQ} = \frac{PQ}{QC}.$$

(Hint: Use similar triangles and a theorem or two from Euclidean geometry.)

c. Use (b) to write $(PQ)^2$ in terms of a, b and z.

d. (Before going on, take a step back and remind yourself of what Khayyam was trying to do!) Combine the equations from parts (a) and (c) to complete your verification that Khayyam's construction is correct.

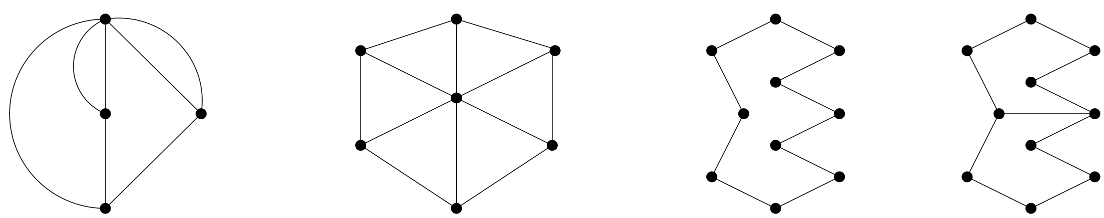
Problem #17 Recall the theorem about when a graph G has an Euler circuit or or Euler path:

- If G has no vertices of odd degree, then it has an Euler circuit.
- If G has two vertices of odd degree, then it has an Euler path but no Euler circuit.
- If G has four or more vertices of odd degree, then it has neither an Euler circuit nor an Euler path.

Obviously there are some missing cases — what if G has one or three vertices of odd degree? In this problem, you'll convince yourself that those cases can't actually happen.

(a) For each of the four graphs G_1, G_2, G_3, G_4 below:
(i) find the degree of each vertex;
(ii) add up the degrees of all the vertices;
(iii) count the edges.

(Suggestion: Keep track of the answers to (ii) and (iii) in a table.)



G_1 G_2 G_3 G_4

(b) What do you notice? Based on the numbers you've written down, make a conjecture about what happens in an arbitrary graph (and, for extra credit, explain why your conjecture is true).

(c) What does your conjecture imply about the number of odd-degree vertices in any graph?

Problem #18 Is it possible to draw a map with ten countries — let's call them Algeria, Australia, Belarus, Bulgaria, Congo, Cyprus, Namibia, Nepal, Pakistan, Portugal — so that each country is contiguous, and every two countries without the same first letter share a common border? (For example, Belarus borders Pakistan, but Congo does not border Cyprus.) Either draw such a map, or explain why it is impossible to do so.

Problem #19 Let $f(x) = \dfrac{\arctan x}{\pi} + \dfrac{1}{2}$ for $x \in \mathbb{R}$.

(a) What is the range of f? (Your answer should be a subset of $\mathbb{R}$.)

(b) Explain why f is a bijection between its domain and range. (You might have to use a little calculus.)

(c) What can you conclude about the cardinalities (i.e., sizes) of the domain and the range of f?

Problem #20 You are the manager of a hotel with infinitely many rooms. One night, a guest wants to check into the hotel, but unfortunately all the rooms are already full. How can you accommodate the new guest? (Hint: You may need to ask other guests to move, but of course you can't actually kick anyone out of the hotel.)

Problem #21 Let $I = [0,1]$ (i.e., the set of real numbers x such that $0 \leq x \leq 1$). For each $x \in I$, let x_i be the i^{th} digit in the decimal expansion of x. For example, if $x = \pi/10 = 0.31415926\ldots$, then $x_1 = 3$, $x_2 = 1$, $x_3 = 4$, $x_4 = 1$, etc. So, in general, the decimal expansion of x is $x_1 x_2 x_3 \ldots$.

(By the way, how do you express 1 in this way? Not a problem — in fact $1 = 0.99999999\ldots$.)

Now, for $x, y \in I$, let $f(x,y)$ be the number you get by "interweaving" the decimal expansions of x and y to get $f(x,y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \cdots$ For example, if

$$x = \pi/10 = 0.31415926\ldots \quad \text{and} \quad y = 8/9 = 0.88888888\ldots,$$

then

$$f(x,y) = 0.3818481858982868\ldots$$

(a) What is the range of f?

(b) Explain why the function f is one-to-one.

(c) Explain why the function f is onto.

(d) Using the conclusions of the previous parts, explain why there are as many points on a single side of a square as there are in the entire square.

Problem #22 Remember that an 0-dimensional simplex is a point, a 1-dimensional simplex is a line segment, a 2-dimensional simplex is a triangle, and a 3-dimensional simplex is a tetrahedron (a.k.a. a triangular pyramid). Let $a(n, k)$ be the number of k-dimensional simplices in an n-dimensional simplex. For example, a triangle (a simplex of dimension 2) has 3 edges (simplices of dimension 1), so $a(2, 1) = 3$.

(#22a) Fill in the rest of the table:

$a(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$				
$n=1$				
$n=2$		3		
$n=3$				

(#22b) These numbers should look familiar. How many 3-dimensional simplices are there in a 7-dimensional simplex?

(#22c) Based on the numbers in the table, make an educated guess at the values of

$$a(7,0) + a(7,1) + a(7,2) + a(7,3) + a(7,4) + a(7,5) + a(7,6) + a(7,7)$$

and

$$a(7,0) - a(7,1) + a(7,2) - a(7,3) + a(7,4) - a(7,5) + a(7,6) - a(7,7).$$

Better yet, make an educated guess and prove it.

Problem #23 We have seen that Euler's formula implies that any planar graph with v vertices and e edges satisfies the inequality $e \leq 3v - 6$. Use this fact to give another solution to the "ten countries" problem above.

Problem #24 First, read the article "Runs in Coin Tossing: Randomness Revealed" by Geoffrey C. Berresford, published in the *College Mathematics Journal*, volume 33, no. 5 (Nov., 2002), pages 391–394. (Your first task is to find the article! KU has an electronic subscription to this journal, so you can access it through the KU Libraries website at lib.ku.edu; start with the "E-journals" link on the left-hand side.) Don't worry if you don't understand every word of the article.

Second, once you've read the article, answer the following question. Here are two sequences of 0's and 1's, both containing 64 digits. One of them was randomly generated by a computer; the other one was made up by Prof. Martin. Make an intelligent guess as to which is which and write a short explanation why, citing the Berresford article.

Sequence A:
1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0

Sequence B:
1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1

Problem #25 Measure the height of the Campanile. Explain the tools you have used, the measurements you have made, and the mathematics behind your answer. Your explanation should make sense to an ancient Greek mathematician (so it is OK to use length, area, similar triangles and the Pythagorean theorem, but not sines and cosines).

Once you are done, find the official height on the KU website (give a citation) and compare your answer to it. They probably won't be the same. This is OK; explain why the two answers might differ.

This should take about a paragraph or two, and you will probably want to include a figure. Your grade will not be based on how close your measurement is to the official height, but rather on your correct use of correct mathematics and your ability to explain your method.

References

[1] David M. Burton, *Burton's History of Mathematics: An Introduction*, 3rd edn., Wm. C. Brown Publishers, 1995.

[2] Underwood Dudley, *Mathematical Cranks*, Mathematical Association of America, 1992.

[3] John J. O'Connor and Edmund F. Robertson, The MacTutor History of Mathematics Archive. Homepage: http://www-groups.dcs.st-and.ac.uk/~history/index.html.

[4] John Stillwell, *Mathematics and its History*, Springer, New York, 1989.