Math 409/410, Spring 2013
Conic Sections

A conic section is a curve that can be formed by slicing a cone with a plane. (Google "conic sections" for tons of pictures.) There are four kinds of conic sections: circles, ellipses, parabolas and hyperbolas (although a circle is a special kind of ellipse).

Nowadays, we know conic sections by their equations. A circle is given by an equation like $x^2 + y^2 = 4$, or parametrized by something like $x = 3\cos t$, $y = 3\sin t$. Parabolas are familiar as the graphs of quadratic polynomials.

The ancient Greeks knew about conic sections, but in a very different way. They didn't have coordinates (which were introduced by Descartes in the 16th century) or sines and cosines (although they did know several trigonometric theorems, just expressed differently). They really did think of them as the curves you get by slicing a cone. We won't do that, but we will look at some purely geometric ways of constructing parabolas and ellipses.

First, we need the idea of a locus, which just means a set of points (plural: loci). When you make a figure in Sketchpad in which point Y depends on point X, we'd like to see the set of all possible points Y. That's a locus. I'll use fancy script letters like $\mathscr{L}$ for loci.

Simple example. Draw a circle and put a fresh point X on it. Put another point Y somewhere else. Draw segment $\overline{XY}$ and construct its midpoint Z. Move X around the circle and watch what happens to Z. What locus does Z trace out?

You can get Sketchpad to construct the locus of all possible locations for Z. Select the points X and Z and select Locus from the Construct menu.

Now leave X and Y alone and move Z around. What happens to the locus?

Another, more exciting example. Draw a circle, put a point X on it, and construct the line N tangent to the circle at X. Put a point Y somewhere else. Draw a line M through Y perpendicular to N, and mark the point Z where M and N meet. Notice that Z depends on M, which depends on N, which depends on X — so ultimately Z depends on X. Move X around and watch what happens to Z. Use Sketchpad to construct the locus $\mathscr{L}$ of all possible Z's. Is this something you have seen before? Observe how the locus $\mathscr{L}$ itself changes as you move Y around.

The conic sections are loci defined in terms of distance. The simplest example is a circle — it is the locus of all points at a fixed distance from a given point (the center of the circle).

Almost as simple is the locus of all points equidistant from a given pair of points A, B. This is, of course, the perpendicular bisector of the line segment $\overline{AB}$ — in particular, it's a line.

What about the locus $\mathscr{L}$ of points equidistant from a point F and a line D? (Those letters will be explained shortly.) This takes more effort to construct. Suppose we put a point X on D; we then want to construct a point P such that (a) of all points on $\mathscr{L}$, X is the closest one to P; and (b) $FX = FP$. If we can do this, then P will be on the locus. For condition (a), we just need the line $\overleftrightarrow{XP}$ to be perpendicular to D, so draw that line and call it m. For condition (b), we need P to be on the perpendicular bisector of $\overline{XP}$, so draw that line and call it n. If we put P at the intersection of lines m and n, then it will be one of the points we're looking for, and by moving X around, we get all possible points P — i.e., the locus $\mathscr{L}$.

Do this in Sketchpad. What does the locus look like? Right — a parabola! By the way, the letters F and D stand for focus and directrix.

How does the parabola change as you move F and D around — in particular, as F is far away from / close to / on / crossing D?

What is the relationship between line n and the parabola?

What do you notice about the figure when P is on D?

Let O be the point on D closest to F. Measure the lengths OX and XP. Using what we know about parabolas, what relationship would you expect? Can you observe this in Sketchpad? (This is a little subtle — it depends on the location of F.)

Here's the usual definition of an ellipse. If F, G are two points (the foci) and r is some real number strictly greater than the distance FG, then the locus

$$\mathscr{L} = \{P \mid FP + GP = r\}$$

is an ellipse. This is a little trickier to construct in Sketchpad, but it can be done.

1. Why does the definition require $r > FG$? To answer this question, you have to think about what $\mathscr{L}$ would look like in the cases $r < FG$ and $r = FG$. In each cases, use our axioms to prove that $\mathscr{L}$ is what you say it is.

2. What does the ellipse look like if r is only a little bit bigger than FG? How does the ellipse change as r gets larger and larger?

3. What if you keep r the same and bring F and G closer together? (If you like creative notation, this is like describing "$\lim_{F \to G} \mathscr{L}$".)

What about the locus $\mathscr{L}$ of points equidistant from a point F and a circle C? For the moment, let's put F in the interior of C.

1. For a point $Y \in \mathscr{L}$, we need to think about which point $X \in C$ is closest to it (so that we know how to calculate the distance from Y to C" makes sense). Where would X be in terms of Y?

2. Now turn this around. If $X \in C$, then which points Y are closer to X than to any other point on C?

3. Which points are equidistant from X and F? (This one, I'll give you for free because it's easy and I want to give it a name. It's the line B that is the perpendicular bisector of segment $\overline{XF}$.)

4. Which point(s) satisfy both conditions 2 and 3?

5. Use Sketchpad to construct Y from X, and then construct the locus $\mathscr{L}$ of all possible Y, as X ranges over the circle C. What does $\mathscr{L}$ look like? How does it compare to the conics we've already seen?

6. Drag the point F around, keeping it in the circle. How does $\mathscr{L}$ change as you move F? What happens as F gets very close to the center of C? What is the relationship between $\mathscr{L}$ and the line B?

7. Now drag the point F outside of C. How does $\mathscr{L}$ change? What happens as F gets very far away from the circle? Does the relationship between $\mathscr{L}$ and B still hold?