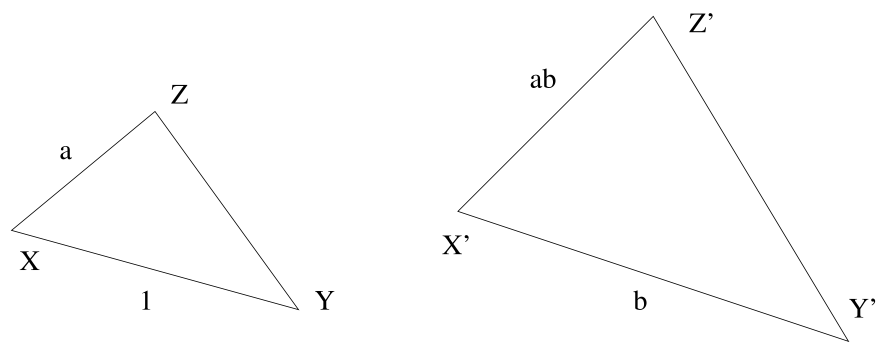
The three classical geometric problems

1 Constructible numbers

Suppose that you are given a line segment of length 1, and the Euclidean tools of compass and (unmarked) straightedge. What other lengths can you construct?

It's easy to construct other integer lengths n, by attaching n copies of your unit length back to back along a line. A little more generally, if you can construct lengths x and y, then you can construct $x+y$ and $x-y$.

What about multiplication and division? The trick is to use similar triangles. Given lengths a and b, you can construct ab as follows. First, draw a triangle ΔXYZ with $XY = 1$ and $XZ = a$. Then, draw a segment $X'Y'$ of length b, and construct a new triangle $\Delta X'Y'Z'$ similar to the first one. Then $X'Z' = ab$.



If instead you start by making $X'Z' = b$, then the new triangle will have $X'Y' = b/a$.

So the set $\mathbb{K}$ of all constructible numbers is closed under the four arithmetic operations of addition, subtraction, multiplication and division.[1] In particular, every rational number is constructible.

But there are certainly non-rational numbers that are constructible. For example, $\sqrt{2}$ can be constructed as the length of the hypotenuse of an isosceles right triangle with sides of length 1. We can then construct segments with lengths like $5\sqrt{2}$, $\sqrt{2}/3-7$, ...

In fact, if k is any constructible number then $\sqrt{k}$ is also constructible. The idea is that

$$\frac{k}{\sqrt{k}} = \frac{\sqrt{k}}{1}$$

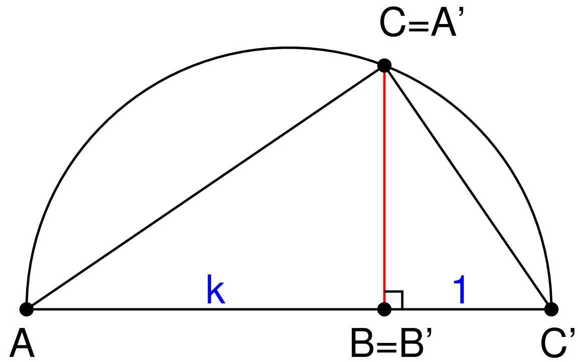
so we'd like to construct a pair of similar triangles ΔABC, $\Delta A'B'C'$ such that

$$\frac{AB}{BC} = \frac{k}{\sqrt{k}} = \frac{\sqrt{k}}{1} = \frac{A'B'}{B'C'}$$

Since both of these triangles are to have sides of length $\sqrt{k}$, it makes sense to arrange the construction so that BC and $A'B'$ are the same segment, and AB has length k and $B'C'$ has length 1.

The following construction solves the problem. Draw a line and mark points $A, B = B', C'$ such that $AB = k$ and $B'C' = 1$. Then draw a circle with AC as a diameter, and drop a perpendicular from B to the circle. The intersection point is both C and A'.

[1] Algebraically, this says exactly that the set $\mathbb{K}$ is a field.



What is the proof that this works? We have a theorem that says $m\angle AXC = 90°$. From this you can deduce that $m\angle BAC = m\angle B'A'C'$ and $m\angle ACB = m\angle A'C'B'$ (details omitted), so by AAA it follows that $\Delta ABC \sim \Delta A'B'C'$. If we let $x = BC = A'B'$, then

$$\frac{AB}{BC} = \frac{k}{x} = \frac{A'B'}{B'C'} = \frac{x}{1}$$

and clearing denominators from $k/x = x$ gives $k = x^2$, or $x = \sqrt{k}$.

This means that $\mathbb{K}$ contains all real numbers that can be formed using the operations $+, -, \times, /, \sqrt{\ }$. For example, if you really want to, you can construct a segment of length

$$\frac{\sqrt{7+2\sqrt{5}} - \sqrt{7-2\sqrt{5}}}{\frac{1}{2} + \sqrt{2013}}$$

using only a straightedge and compass.

So the question arises: What exactly is the set $\mathbb{K}$? That is, which real numbers can be constructed as the lengths of line segments?

2 The three classical problems

There are three famous problems of classical geometry that the Greeks were unable to solve (for good reason). They were:

1. **Squaring the circle.** Given a circle, construct a square of the same area.
2. **Trisecting the angle.** Given an arbitrary angle of measure α, construct an angle of measure $\alpha/3$.
3. **Squaring the circle.** Given a cube, construct a new cube whose volume is double that of the first cube.

All these problems can be rephrased in terms of constructible numbers. A circle of radius 1 has area π, so we can square the circle iff we can construct $\sqrt{\pi}$. A cube Q of side length 1 has volume 1; to double it, we'd need to construct a line segment of length $\sqrt[3]{2}$. Finally, it turns out that trisecting an arbitrary angle is equivalent to being able to construct a root of a certain cubic equation.

It turns out that all these things are impossible — the set of constructible numbers is known not to include transcendentals (like $\sqrt{\pi}$) or things like cube roots. In fact, $\mathbb{K}$ contains exactly the real numbers that can be formed using the operations $+, -, \times, /, \sqrt{\ }$, and no others.

To prove this, you need analytic geometry — the idea that curves can be described by equations, which Descartes came up with around 1600. (The Greeks thought of geometry and arithmetic as very separate disciplines.) Roughly, the idea is that lines and circles are described by linear and quadratic equations respectively, and every constructible number has to satisfy either a linear or quadratic equation in previously constructed numbers. Cube roots don't do this, nor do transcendental numbers like π.

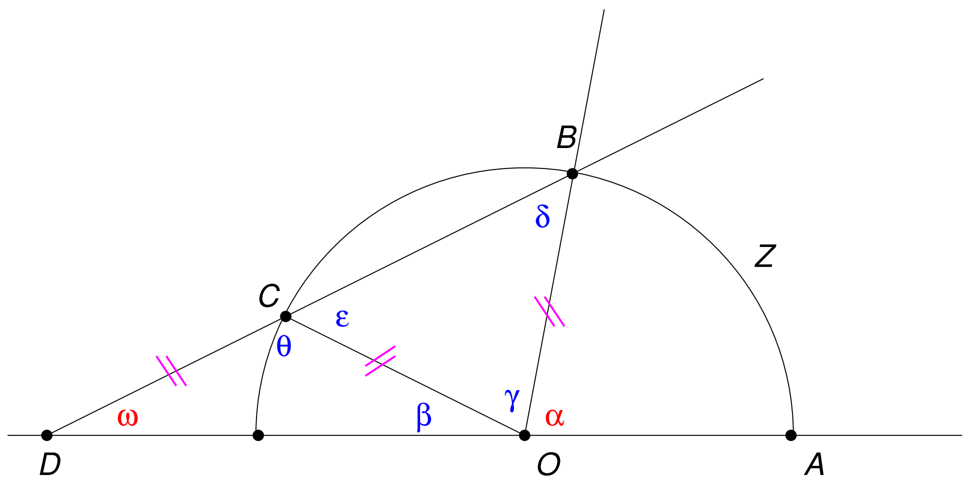
3 How to trisect the angle

The following solution of the angle trisection problem is attributed to Archimedes. While the construction works, it is not Euclidean because it uses a new tool: a straightedge with two points marked on it. (The Greek mathematicians called this kind of construction neusis, and disapproved of it.)

Let $\angle AOB$ be the angle to be trisected. Draw a circle Z centered at O, and call the radius of the circle r. We might as well assume that A and B both lie on Z.

Now, here's the non-Euclidean step. Mark two points C, D on your straightedge at distance r. Then, move the straightedge so that (i) C lies on $\overleftrightarrow{OA}$; (ii) D lies on the circle Z; and (iii) the line of the straightedge goes through B.

(You can simulate this in Sketchpad. Start by drawing a circle Z centered at O, putting points A, B on O, and drawing the angle $\angle AOB$. Construct points C and D satisfying conditions (i) and (ii) — this is not too bad. Then draw $\overleftrightarrow{CD}$ and wiggle C and D around until condition (iii) is satisfied. This is the neusis part. Of course, if you wiggle C and D around more then condition (iii) will fail and it will mess up the sketch.)



Labelling the angles $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \omega$ as shown[2], it can now be proven that $\omega = \alpha/3$. The details are left as an exercise, but the only facts you need are the following:

1. If adding two or more angles makes a straight line, then their sum is $180°$.
2. In every triangle, its three angles sum to $180°$.
3. Thales' Theorem: the base angles of an isosceles triangle are equal. (That is, if $PQ = PR$ then $m\angle PQR = m\angle PRQ$.)

Each of these facts can be applied twice in the figure to give a total of six equations on $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \omega$, and then you can use algebra to deduce that $\omega = \alpha/3$.

So the angle can be trisected — if you allow yourself the right tools. As with all good mathematical discoveries, this observation opens up a whole host of interesting questions: What tools are needed to construct a segment of length $\sqrt[3]{2}$? What tools are needed to construct a segment of length π? Given a particular toolbox, how do you describe the set of numbers constructible from it?

[2]Note: θ is supposed to be $\angle OCD$. Don't let the arc from A to C confuse you.